

# CENTRAL LIMIT THEOREM FOR LINEAR SPECTRAL STATISTICS OF LARGE DIMENSIONAL SEPARABLE SAMPLE COVARIANCE MATRICES

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**ABSTRACT.** Suppose that  $\mathbf{X}_n = (x_{jk})$  is  $N \times n$  whose elements are independent real variables with mean zero, variance 1 and the fourth moment equal to three. The separable sample covariance matrix is defined as  $\mathbf{B}_n = \frac{1}{N} \mathbf{T}_{2n}^{1/2} \mathbf{X}_n \mathbf{T}_{1n} \mathbf{X}_n' \mathbf{T}_{2n}^{1/2}$  where  $\mathbf{T}_{1n}$  is a symmetric matrix and  $\mathbf{T}_{2n}^{1/2}$  is a symmetric square root of the nonnegative definite symmetric matrix  $\mathbf{T}_{2n}$ . Its linear spectral statistics (LSS) are shown to have Gaussian limits when  $n/N$  approaches a positive constant.

**Keywords:** Central limit theorem, General sample covariance matrix, Large dimension, Linear spectral statistics, Random matrix theory.

## 1. INTRODUCTION

The sample covariance matrix is one of the most commonly studied random matrices in Random Matrix Theory, which can be traced back to Wishart (1928) (see [20]). It plays an important role in multivariate analysis because many statistics in traditional multivariate statistical analysis (e.g., principle component analysis, factor analysis and multivariate regression analysis) can be written as functionals of the eigenvalues of sample covariance matrices.

Large dimensional data now appear in various fields such as finance and genetic experiments due to different reasons. To deal with such large-dimensional data, a new area in asymptotic statistics has been developed where the data dimension  $p$  is no more fixed but tends to infinity together with the sample size  $n$ . The random matrices proves to be a powerful tool for such large dimensional statistical problems. One may refer to the latest book in this area by J. F. Yao, S. R. Zheng and Z. D. Bai (2015), the recent work by Ledoit and Wolf (2004) and Jiang and Yang (2013).

So far, most work focus on the sample covariance matrices of the form

$$\mathbf{S}_n = \frac{1}{N} \mathbf{T}_n^{1/2} \mathbf{X}_n \mathbf{X}_n' \mathbf{T}_n^{1/2}$$

where  $\mathbf{X}_n$  is a  $N \times n$  matrix with independent entries and  $\mathbf{T}_n$  is a nonnegative definite symmetric matrix. As we know  $\mathbf{S}_n$  can be viewed as a sample covariance matrix formed from  $n$  samples of the random vector  $\mathbf{T}_n^{1/2} \mathbf{x}_1$  (where  $\mathbf{x}_1$  denotes the

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1991 *Mathematics Subject Classification.* Primary 15B52, 60F15, 62E20; Secondary 60F17.

H. Q. Li was partially supported by China Scholarship Council;

first column of  $\mathbf{X}_n$ , which has population covariance matrix  $\mathbf{T}_n$ . Much work has been done on the central limit theorem (CLT) for linear eigenvalues statistics of  $\mathbf{S}_n$  under different assumptions. Among others we mention [1, 10, 14, 16, 18]. One of the key features of the above sample covariance matrices  $\mathbf{S}_n$  is that the sample are independent. As far as we know there is no CLT available for the sample covariance matrices generated from the dependent sample.

In view of the above we consider a kind of general sample covariance matrices

$$(1.1) \quad \mathbf{B}_n = \frac{1}{N} \mathbf{T}_{2n}^{1/2} \mathbf{X}_n \mathbf{T}_{1n} \mathbf{X}_n' \mathbf{T}_{2n}^{1/2},$$

where  $\mathbf{T}_{2n}$  is  $N \times N$  nonnegative definite symmetric matrix and  $\mathbf{T}_{1n}$  is  $n \times n$  symmetric. This model finds applications in the diverse fields including spatio-temporal statistics, wireless communications and econometrics. For example, the data matrix can be represented as

$$(1.2) \quad \mathbf{Y}_n = \mathbf{T}_{2n}^{1/2} \mathbf{X}_n \mathbf{T}_{1n}^{1/2}$$

if  $\mathbf{T}_{1n}$  is nonnegative definite symmetric. Denote by  $\text{vec}(\mathbf{Y}_n)$  the vector operator that stacks the columns of  $\mathbf{Y}_n$  into a column vector. This model is referred to as a separable covariance model because the covariance of  $\text{vec}(\mathbf{Y}_n)$  is the Kronecker product of  $\mathbf{T}_{1n}$  and  $\mathbf{T}_{2n}$ . The rows of the data matrix  $\mathbf{Y}_n$  correspond to indices of spatial locations and the column indices correspond to points in time in the field of spatio-temporal statistics. This covariance structure implies that the entries of  $\mathbf{Y}_n$  are correlated in time (column), but the pattern of temporal correlation does not change with location (row). One may see [17] and the references therein.

In econometrics, when determining the number of factors in the approximate factor models [15] assumes that the idiosyncratic components of the data is of the form  $\mathbf{Y}_n$ . This allows the idiosyncratic terms to be non-trivially correlated both cross-sectionally and over time. The cross-sectional correlation is caused by matrix  $\mathbf{T}_{2n}^{1/2}$  linearly combining different rows of  $\mathbf{X}_n$ , whereas the correlation over time is caused by matrix  $\mathbf{T}_{1n}^{1/2}$  linearly combining different columns of  $\mathbf{X}_n$ .

Another motivation of considering the sample covariance matrices  $\mathbf{B}_n$  is the matrix data. Matrix observations are becoming increasingly available due to the rapid advance in the information technology. For example, images are routinely stored as pixel by pixel data; agricultural exports can be represented via matrices, one for each year, with rows denoting for example different regions and columns different produces; the gene expression of a single subject can be organized as a matrix with the rows for tissue types and the columns for genes. There is an abundance of data that can be characterized as matrix variates in food sciences and chemometrics. In general, the sample covariance matrix of the matrix data is

$$\frac{1}{nm} \sum_{k=1}^m \mathbf{Y}_{nk} \mathbf{Y}_{nk}',$$

where  $\mathbf{Y}_{nk}, k = 1, \dots, m$  are  $n \times N$  matrix data. Some papers argued that for many matrix variates, it is more appropriate to assume that

$$\text{Cov}(\text{vec}(\mathbf{Y}_{nk})) = \mathbf{T}_{1n} \otimes \mathbf{T}_{2n}.$$

One may refer to [12] and the references therein. Here, the matrix data  $\mathbf{Y}_n$  defined in (1.2) is just one matrix observation for simplicity. Moreover, write

$$\frac{1}{nm} \sum_{k=1}^m \mathbf{Y}_{nk} \mathbf{Y}_{nk}' = \frac{1}{nm} \mathbf{T}_{2n}^{1/2} (\mathbf{X}_{n1}, \dots, \mathbf{X}_{nm}) (\mathbf{I}_m \otimes \mathbf{T}_{1n}) (\mathbf{X}_{n1}, \dots, \mathbf{X}_{nm})' \mathbf{T}_{2n}^{1/2}.$$

Let  $\tilde{\mathbf{T}}_{1n} = \mathbf{I}_m \otimes \mathbf{T}_{1n}$  and  $\tilde{\mathbf{X}}_n = (\mathbf{X}_{n1}, \dots, \mathbf{X}_{nm})$ . Then

$$\frac{1}{nm} \sum_{k=1}^m \mathbf{Y}_{nk} \mathbf{Y}_{nk}' = \frac{N}{nm} \frac{1}{N} \mathbf{T}_{2n}^{1/2} \tilde{\mathbf{X}}_n \tilde{\mathbf{T}}_{1n} \tilde{\mathbf{X}}_n' \mathbf{T}_{2n}^{1/2}$$

which has the same form as (1.1) if  $nm$  and  $N$  are of the same order.

For any hermitian matrix  $\mathbf{A}$  of size  $n \times n$  its empirical spectral distribution (ESD) is defined by

$$F^{\mathbf{A}}(x) = \frac{1}{n} \sum_{j=1}^n I(\lambda_j \leq x),$$

where  $\{\lambda_j\}$  are eigenvalues of  $\mathbf{A}$ . For  $\mathbf{B}_n$  defined in (1.1), a number of papers ([5] and [23]) investigated its empirical spectral distribution  $F_{\mathbf{B}_n}$  and the weakest assumption is given in [23], which is specified below. To characterize its limit define the Stieltjes transform of any distribution function  $F^{\mathbf{A}}(x)$  to be

$$m_{F^{\mathbf{A}}}(z) = \int \frac{1}{x - z} dF^{\mathbf{A}}(x) = \frac{1}{n} \text{tr}(\mathbf{A} - z\mathbf{I})^{-1}, \quad z \in \mathbb{C}^+.$$

Throughout the paper we make the following assumption.

**Condition 1.1.**

- (i)  $\mathbf{X}_n = (x_{jl})$  is  $N \times n$  consisting of independent real random variables with  $\mathbb{E}x_{jl} = 0, \mathbb{E}x_{jl}^2 = 1$ , satisfying for each  $\delta > 0$ , as  $n \rightarrow \infty$

$$\frac{1}{\delta^2 n N} \sum_{j,l} \mathbb{E} \left( x_{jl}^2 I(|x_{jl}| > \delta \sqrt{n}) \right) \rightarrow 0.$$

- (ii)  $\mathbf{T}_{1n}$  is  $n \times n$  real symmetric matrix (without loss of generality, we assume that  $\mathbf{T}_{1n}$  is not semi-negative definite) and  $\mathbf{T}_{2n}$  is  $N \times N$  nonnegative definite real symmetric matrix.
- (iii) With probability 1, as  $n \rightarrow \infty$ , the empirical spectral distributions of  $\mathbf{T}_{1n}$  and  $\mathbf{T}_{2n}$ , denoted by  $H_{1n}$  and  $H_{2n}$  respectively, converge weakly to two probability functions  $H_1$  and  $H_2$ , respectively.
- (iv)  $N = N(n)$  with  $n/N \rightarrow c > 0$ .
- (v)  $\mathbf{X}_n, \mathbf{T}_{1n}, \mathbf{T}_{2n}$  are independent.

L. X. Zhang [23] establishes the following conclusion under Condition 1.1. For  $\mathbf{B}_n$  defined in (1.1), with probability 1, as  $n \rightarrow \infty$ , the ESD of  $\mathbf{B}_n$  converges weakly to a non-random probability distribution function  $F$  for which if  $H_1 = 1_{[0,\infty)}$  or  $H_2 = 1_{[0,\infty)}$ , then  $F = 1_{[0,\infty)}$ ; otherwise the Stieltjes transform  $m(z)$  of  $F$  is determined by the following system of equations (1.3), where for each  $z \in \mathbb{C}^+$ ,

$$(1.3) \quad \begin{cases} s(z) = -z^{-1}(1-c) - z^{-1}c \int \frac{1}{1+q(z)x} dH_1(x) \\ s(z) = -z^{-1} \int \frac{1}{1+p(z)y} dH_2(y) \\ s(z) = -z^{-1} - p(z)q(z). \end{cases}$$

Then, the Stieltjes transform  $m(z)$  of  $F$ , together with the two other functions, denoted by  $g_1(z)$  and  $g_2(z)$ ,  $(m(z), g_1(z), g_2(z))$  is the unique solution to (1.3) in the set

$$U = \left\{ (s(z), p(z), q(z)) : \Im s(z) > 0, \Im(zp(z)) > 0, \Im q(z) > 0 \right\}$$

where  $\Im h(z)$  stands for the imaginary part of  $h(z)$ . Denote  $\underline{\mathbf{B}}_n = \frac{1}{N} \mathbf{T}_{1n} \mathbf{X}_n' \mathbf{T}_{2n} \mathbf{X}_n$ . Then we have the following relationship between the empirical distributions of  $\mathbf{B}_n$  and  $\underline{\mathbf{B}}_n$

$$F^{\mathbf{B}_n}(x) = c_n F^{\underline{\mathbf{B}}_n}(x) + (1 - c_n) I_{[0,\infty)}(x),$$

and hence

$$(1.4) \quad m_n(z) = c_n \underline{m}_n(z) + z^{-1}(c_n - 1).$$

where  $c_n = n/N$ ,  $m_n(z) = m_{F^{\mathbf{B}_n}}(z)$  and  $\underline{m}_n(z) = m_{F^{\underline{\mathbf{B}}_n}}(z)$ . Denote by  $\underline{F}$  the limiting distribution of  $F^{\underline{\mathbf{B}}_n}$ . Then  $F$  and  $\underline{F}$  must satisfy

$$F(x) = c \underline{F}(x) + (1 - c) I_{[0,\infty)}(x),$$

and

$$(1.5) \quad m(z) = c \underline{m}(z) - z^{-1}(1 - c)$$

where  $\underline{m}(z) = m_{\underline{F}}(z)$ . If we let  $F^{c, H_1, H_2}$  denote  $F$ , then  $F^{c_n, H_{1n}, H_{2n}}$  is obtained from  $F^{c, H_1, H_2}$  with  $c, H_1, H_2$  replaced by  $c_n, H_{1n}, H_{2n}$  respectively. Let  $m_n^0(z) = m_{F^{c_n, H_{1n}, H_{2n}}}(z)$  for simplicity. Moreover  $g_{1n}^0(z)$  and  $g_{2n}^0(z)$  are similarly obtained from  $g_1(z)$  and  $g_2(z)$  respectively. Then  $(m_n^0(z), g_{1n}^0(z), g_{2n}^0(z))$  satisfies the equations (1.3). In other words

$$(1.6) \quad \underline{m}_n^0(z) = -z^{-1} \int \frac{1}{1 + g_{2n}^0(z)x} dH_{1n}(x)$$

$$(1.7) \quad m_n^0(z) = -z^{-1} \int \frac{1}{1 + g_{1n}^0(z)y} dH_{2n}(y)$$

$$(1.8) \quad m_n^0(z) = -z^{-1} - g_{1n}^0(z)g_{2n}^0(z).$$

Furthermore,

$$(1.9) \quad zg_{1n}^0(z) = -c_n \int \frac{x}{1 + g_{2n}^0(z)x} dH_{1n}(x)$$

$$(1.10) \quad zg_{2n}^0(z) = - \int \frac{y}{1 + g_{1n}^0(z)y} dH_{2n}(y).$$

[6] further investigated the limiting spectral measure of  $\mathbf{B}_n$  and [17] proved that no eigenvalues exist outside the support of limiting empirical spectral distribution of  $\mathbf{B}_n$ . But [17] required  $\mathbf{T}_{2n}$  in  $\mathbf{B}_n$  to be diagonal (with positive diagonal entries). It is well known that many important statistics in multivariate analysis can be written as functionals of the ESD of some random matrices. In view of this the aim of this paper is to establish the central limit theorem for linear spectral statistics (LSS) of  $\mathbf{B}_n$ . LSS of general sample covariance matrices are quantities of the form

$$\frac{1}{N} \sum_{j=1}^N f(\lambda_j^{\mathbf{B}_n}) = \int f(x) dF^{\mathbf{B}_n}(x)$$

where  $f$  is some continuous and bounded real function on  $(-\infty, \infty)$ .

This paper is organized as follows. Section 2 establishes the main result about the CLT for LSS of  $\mathbf{B}_n$ . By the Stieltjes transform method, we complete the proof of theorem when the entries of matrix are Gaussian variables in Section 3. Section 4 extends the result from the Gaussian case to the general case through comparing their characteristic functions. Some useful lemmas are listed in Appendix.

## 2. MAIN RESULT

Define

$$G_n(x) = N\left(F^{\mathbf{B}_n}(x) - F^{c_n, H_{1n}, H_{2n}}\right).$$

The main result is stated in the following theorem.

**Theorem 2.1.** *Denote by  $s_1 \geq \dots \geq s_n$  ( $s_1 > 0$ ) the eigenvalues of  $\mathbf{T}_{1n}$ . Let  $f_1, \dots, f_k$  be functions on  $\mathbb{R}$  analytic on an open interval containing*

$$(2.1) \quad \left[ \liminf_n s_n \left( \lambda_{\min}^{\mathbf{T}_{2n}} I_{(0,1)}(c) (1 - \sqrt{c})^2 I(s_n \geq 0) + \lambda_{\max}^{\mathbf{T}_{2n}} (1 + \sqrt{c})^2 I(s_n < 0) \right), \right. \\ \left. \limsup_n s_1 \left( \lambda_{\max}^{\mathbf{T}_{2n}} (1 + \sqrt{c})^2 \right) \right].$$

*In addition to Condition 1.1 we further suppose that  $\text{Ex}_{jl}^4 = 3$  and for each  $\delta > 0$ , as  $n \rightarrow \infty$*

$$\frac{1}{\delta^4 n^2} \sum_{j,l} \mathbb{E} \left( x_{jl}^4 I(|x_{jl}| > \delta \sqrt{n}) \right) \rightarrow 0.$$

Also suppose that the spectral norms of  $\mathbf{T}_{1n}$  and  $\mathbf{T}_{2n}$  are both bounded in  $n$ . Then

$$\left( \int f_1(x) dG_n(x), \dots, \int f_k(x) dG_n(x) \right)$$

converges weakly to a Gaussian vector  $(X_{f_1}, \dots, X_{f_k})$  with mean

$$(2.2) \quad \mathbb{E}X_f = -\frac{1}{2\pi i} \oint_C f(z) \left\{ (d_1(z) - d_2(z)) \left\{ 1 - z^{-1} \left[ \int \frac{x}{(1 + xg_2(z))^2} dH_1(x) \right]^{-1} \right. \right. \\ \left. \left. \times \int \frac{x^2}{(1 + xg_2(z))^2} dH_1(x) \int \frac{t}{(1 + g_1(z)t)^2} dH_2(t) \right\}^{-1} \right\} dz$$

and covariance function

$$(2.3) \quad \text{Cov}(X_f, X_g) = -\frac{1}{2\pi^2} \oint_{C_1} \oint_{C_2} \frac{\partial^2}{\partial z_2 \partial z_1} \int_0^{f(z_1, z_2)} \frac{1}{1-z} dz dz_1 dz_2$$

where  $f, g \in \{f_1, \dots, f_k\}$ . Here

$$d_1(z) = -cz^{-3} \int \frac{x^2}{(1 + xg_2(z))^2} dH_1(x) \int \frac{t^2}{(g_1(z)t + 1)^3} dH_2(t) \\ \times \left[ 1 - cz^{-2} \int \frac{x^2}{(1 + xg_2(z))^2} dH_1(x) \int \frac{t^2}{(g_1(z)t + 1)^2} dH_2(t) \right]^{-1} \\ - cz^{-4} \int \frac{x^3}{(1 + xg_2(z))^3} dH_1(x) \int \frac{t}{(g_1(z)t + 1)^2} dH_2(t) \int \frac{t^2}{(g_1(z)t + 1)^2} dH_2(t) \\ \times \left[ 1 - cz^{-2} \int \frac{x^2}{(1 + xg_2(z))^2} dH_1(x) \int \frac{t^2}{(g_1(z)t + 1)^2} dH_2(t) \right]^{-1},$$

$$d_2(z) = -cz^{-4} \int \frac{x^2}{(1 + xg_2(z))^3} dH_1(x) \int \frac{t^2}{(g_1(z)t + 1)^2} dH_2(t) \\ \times \left[ \int \frac{x}{(1 + xg_2(z))^2} dH_1(x) \right]^{-1} \int \frac{x^2}{(1 + xg_2(z))^2} dH_1(x) \int \frac{t}{(1 + g_1(z)t)^2} dH_2(t) \\ \times \left[ 1 - cz^{-2} \int \frac{x^2}{(1 + xg_2(z))^2} dH_1(x) \int \frac{t^2}{(g_1(z)t + 1)^2} dH_2(t) \right]^{-1},$$

and

$$f(z_1, z_2) = \frac{1}{z_1 z_2} \frac{z_1 g_1(z_1) - z_2 g_1(z_2)}{g_2(z_1) - g_2(z_2)} \frac{z_1 g_2(z_1) - z_2 g_2(z_2)}{g_1(z_1) - g_1(z_2)}.$$

The contours in (2.2) and (2.3) (two contours in (2.3), which we may assume to be nonoverlapping) are closed and are taken in the positive direction in the complex plane, each enclosing the support of  $F^{c, H_1, H_2}$ .

**Remark 2.2.** It is worth mentioning that our result is consistent with that in [1]. We distinguish two cases to show the consistency according to whether  $\mathbf{T}_{2n}$  or  $\mathbf{T}_{1n}$  reduces to the identity matrix.

When  $\mathbf{T}_{2n} = \mathbf{I}$  and  $\mathbf{T}_{1n}$  is a nonnegative definite symmetric matrix,  $\mathbf{B}_n = \frac{1}{N}\mathbf{X}_n\mathbf{T}_{1n}\mathbf{X}_n'$ . Then (1.3) is transformed into

$$\begin{cases} \underline{m}(z) = -z^{-1} \int \frac{1}{1+\underline{m}(z)x} dH_1(x) \\ g_1(z) = -\frac{1}{z\underline{m}(z)} - 1 \\ g_2(z) = \underline{m}(z). \end{cases}$$

It follows that

$$\text{EX}_f = -\frac{1}{2\pi i} \oint_C f(z) \int \frac{cm(z)^3 x^2}{(1+xm(z))^3} dH_1(x) \left\{ 1 - \int \frac{cm(z)^2 x^2}{(1+xm(z))^2} dH_1(x) \right\}^{-2} dz$$

and

$$f(z_1, z_2) = 1 + \frac{m(z_1)m(z_2)(z_1 - z_2)}{m(z_2) - m(z_1)}.$$

These are the same as those in [1].

If  $\mathbf{T}_{1n} = \mathbf{I}$  then  $\mathbf{B}_n = \frac{1}{N}\mathbf{T}_{2n}^{1/2}\mathbf{X}_n\mathbf{X}_n'\mathbf{T}_{2n}^{1/2}$ . Let  $\widetilde{\mathbf{B}}_n = \frac{1}{n}\mathbf{T}_{2n}^{1/2}\mathbf{X}_n\mathbf{X}_n'\mathbf{T}_{2n}^{1/2}$  and  $\underline{\widetilde{\mathbf{B}}}_n = \frac{1}{n}\mathbf{X}_n'\mathbf{T}_{2n}\mathbf{X}_n$ . We use  $\widetilde{m}_n(z)$  and  $\underline{\widetilde{m}}_n(z)$  to denote the Stieltjes transforms of  $F^{\widetilde{\mathbf{B}}_n}$  and  $F^{\underline{\widetilde{\mathbf{B}}}_n}$  respectively. Denote by  $\widetilde{F}^{c^{-1}, H_2}$  the limiting distribution of  $\widetilde{F}^{\widetilde{\mathbf{B}}_n}$ . Moreover  $\widetilde{F}^{c_n^{-1}, H_{2n}}$  is obtained from  $\widetilde{F}^{c^{-1}, H_2}$  with  $c, H_2$  replaced by  $c_n, H_{2n}$  respectively. Let  $\widetilde{m}(z) = \lim_{n \rightarrow \infty} \widetilde{m}_n(z)$ ,  $\underline{\widetilde{m}}(z) = \lim_{n \rightarrow \infty} \underline{\widetilde{m}}_n(z)$  and  $\underline{\widetilde{m}}_n^0(z) = m_{\widetilde{F}^{c_n^{-1}, H_{2n}}}(z)$ . Due to (2.7) below we only need to consider the limiting distribution of  $\widetilde{M}_n(z) = N[\widetilde{m}_n(z) - \underline{\widetilde{m}}_n^0(z)]$ . Firstly, (1.3) becomes

$$\begin{cases} m(z) = -z^{-1} \int \frac{1}{1+\underline{m}(z)x} dH_2(x) \\ g_1(z) = \underline{m}(z) \\ g_2(z) = -\frac{1}{z\underline{m}(z)} - 1 \end{cases}.$$

By Lemma 2.3 below and the above equations, we have

$$(2.4) \quad \text{EM}(z) = \int \frac{\underline{m}(z)^3 x^2}{(1+c\underline{m}(z))^3} dH_2(x) \left\{ 1 - \int \frac{\underline{m}(z)^2 x^2}{(1+c\underline{m}(z))^2} dH_2(x) \right\}^{-2}$$

and

$$f(z_1, z_2) = 1 + \frac{\underline{m}(z_1)\underline{m}(z_2)(z_1 - z_2)}{\underline{m}(z_2) - \underline{m}(z_1)}.$$

Note that  $\mathbf{B}_n = c_n \widetilde{\mathbf{B}}_n$ . It can be verified that

$$\underline{\widetilde{m}}_n(z/c_n) = c_n \underline{\widetilde{m}}_n(z)$$

and

$$M_n(z) = c_n^{-1} \widetilde{M}_n(z/c_n).$$

These imply that

$$(2.5) \quad \underline{\widetilde{m}}(z/c) = c \underline{\widetilde{m}}(z)$$

and

$$(2.6) \quad M(z) = c^{-1} \widetilde{M}(z/c)$$

where  $\widetilde{M}(z)$  is a two-dimensional Gaussian process, the limit of weak convergence of  $\widetilde{M}_n(z)$ . Plugging (2.5) and (2.6) into (2.4), one has

$$E\widetilde{M}(z/c) = c^{-1} \int \frac{\underline{\widetilde{m}}(z/c)^3 x^2}{(1 + x \underline{\widetilde{m}}(z/c))^3} dH_2(x) \left\{ 1 - c^{-1} \int \frac{\underline{\widetilde{m}}(z/c)^2 x^2}{(1 + x \underline{\widetilde{m}}(z/c))^2} dH_2(x) \right\}^{-2}$$

and

$$f(z_1, z_2) = 1 + \frac{\underline{\widetilde{m}}(z_1/c) \underline{\widetilde{m}}(z_2/c) (z_1/c - z_2/c)}{\underline{\widetilde{m}}(z_2/c) - \underline{\widetilde{m}}(z_1/c)}.$$

Hence the expectation and covariance are the same as those in Bai and Siverstein (2004).

By Cauchy's formula

$$(2.7) \quad \int f(x) dG(x) = -\frac{1}{2\pi i} \oint f(z) m_G(z) dz$$

where  $G$  is a cumulative distribution function (c.d.f.) and  $f$  is analytic on an open set containing the support of  $G$ . The complex integral on the right-hand side is over any positively oriented contour enclosing the support of  $G$  and on which  $f$  is analytic. Hence, the proof of Theorem 2.1 relies on establishing limiting results on

$$M_n(z) = N \left[ m_n(z) - m_n^0(z) \right].$$

The contour  $C$  is defined as follows.

By Condition 1.1, we may suppose  $\max \{ \|\mathbf{T}_{1n}\|, \|\mathbf{T}_{2n}\| \} \leq \tau$ . Let  $v_0$  be any positive number. Let  $x_r$  be any positive number if the right end point of interval (2.1) is zero. Otherwise choose

$$x_r \in (\limsup_n s_1 \lambda_{\max}^{\mathbf{T}_{2n}} (1 + \sqrt{c})^2, \infty).$$

Let  $x_l$  be any negative number if the left end point of interval (2.1) is zero. Otherwise choose

$$x_l \in \begin{cases} (0, \liminf_n s_n \lambda_{\min}^{\mathbf{T}_{2n}} I_{(0,1)}(c) (1 - \sqrt{c})^2), & \text{if } \liminf_n s_n \lambda_{\min}^{\mathbf{T}_{2n}} I_{(0,1)}(c) > 0, \\ (-\infty, \liminf_n s_n \lambda_{\max}^{\mathbf{T}_{2n}} (1 + \sqrt{c})^2), & \text{if } \liminf_n s_n \lambda_{\min}^{\mathbf{T}_{2n}} I_{(0,1)}(c) \leq 0. \end{cases}$$



Let

$$C_u = \{x + iv_0 : x \in [x_l, x_r]\}.$$

Define the contour  $C$

$$C = \{x_l + iv : v \in [0, v_0]\} \cup C_u \cup \{x_r + iv : v \in [0, v_0]\}.$$

To avoid dealing with the small  $\Im z$ , we truncate  $M_n(z)$  on a contour  $C$  of the complex plane. We define now the subsets  $C_n$  of  $C$  on which  $M_n(\cdot)$  agrees with  $\widehat{M}_n(\cdot)$ . Choose sequence  $\{\varepsilon_n\}$  decreasing to zero satisfying for some  $\alpha \in (0, 1)$

$$\varepsilon_n \geq n^{-\alpha}.$$

Let

$$C_l = \{x_l + iv : v \in [n^{-1}\varepsilon_n, v_0]\} \quad \text{and} \quad C_r = \{x_r + iv : v \in [n^{-1}\varepsilon_n, v_0]\}.$$

Then  $C_n = C_l \cup C_u \cup C_r$ . For  $z = x + iv$ , the process  $\widehat{M}_n(\cdot)$  can now be defined as

$$(2.8) \quad \widehat{M}_n(\cdot) = \begin{cases} M_n(z), & \text{for } z \in C_n, \\ M_n(x_l + in^{-1}\varepsilon_n), & \text{for } x = x_l, v \in [0, n^{-1}\varepsilon_n], \\ M_n(x_r + in^{-1}\varepsilon_n), & \text{for } x = x_r, v \in [0, n^{-1}\varepsilon_n]. \end{cases}$$

The central limit theorem of  $\widehat{M}_n(z)$  is specified below.

**Lemma 2.3.** *Under the conditions of Theorem 2.1,  $\widehat{M}_n(z)$  converges weakly to a two-dimensional Gaussian process  $M(\cdot)$  satisfying for  $z \in C$*

$$\begin{aligned} EM(z) &= (d_1(z) - d_2(z)) \left\{ 1 - z^{-1} \left[ \int \frac{x}{(1 + xg_2(z))^2} dH_1(x) \right]^{-1} \right. \\ &\quad \left. \times \int \frac{x^2}{(1 + xg_2(z))^2} dH_1(x) \int \frac{t}{(1 + g_1(z)t)^2} dH_2(t) \right\}^{-1} \end{aligned}$$

and for  $z_1, z_2 \in C \cup \overline{C}$  with  $\overline{C} = \{\bar{z} : z \in C\}$ ,

$$\text{Cov}(M(z_1), M(z_2)) = 2 \frac{\partial^2}{\partial z_2 \partial z_1} \int_0^{f(z_1, z_2)} \frac{1}{1 - z} dz.$$

*Proof of Theorem 2.1.* From [22] and [3], we conclude that

$$(2.9) \quad \lambda_{\max} \left( \frac{1}{N} \mathbf{X}'_n \mathbf{X}_n \right) \rightarrow (1 + \sqrt{c})^2 \quad \text{a.s.}$$

and

$$\lambda_{\min} \left( \frac{1}{N} \mathbf{X}'_n \mathbf{X}_n \right) \rightarrow (1 - \sqrt{c})^2 \quad \text{a.s.}$$

The upper and lower bounds of the extreme eigenvalues of  $\mathbf{B}_n$  depends largely on the signs of  $s_1$  and  $s_n$ . Since  $s_1 > 0$ , we have

$$\lambda_{\max}(\mathbf{B}_n) \leq s_1 \lambda_{\max}^{\mathbf{T}_{2n}} \lambda_{\max} \left( \frac{1}{N} \mathbf{X}'_n \mathbf{X}_n \right) \leq s_1 \lambda_{\max}^{\mathbf{T}_{2n}} (1 + \sqrt{c})^2 \quad \text{a.s.}$$

If  $s_n > 0$ , then we have

$$\lambda_{\min}(\mathbf{B}_n) \geq s_n \lambda_{\min}^{\mathbf{T}_{2n}} I_{(0,1)}(c) \lambda_{\min} \left( \frac{1}{N} \mathbf{X}'_n \mathbf{X}_n \right) \geq s_n \lambda_{\min}^{\mathbf{T}_{2n}} I_{(0,1)}(c) (1 - \sqrt{c})^2 \quad \text{a.s.}$$

Otherwise, we get

$$\lambda_{\min}(\mathbf{B}_n) \geq s_n \lambda_{\max}^{\mathbf{T}_{2n}} \lambda_{\max} \left( \frac{1}{N} \mathbf{X}'_n \mathbf{X}_n \right) \geq s_n \lambda_{\max}^{\mathbf{T}_{2n}} (1 + \sqrt{c})^2 \quad \text{a.s.}$$

Combining the definitions of  $x_l$ ,  $x_r$ , we find with probability 1

$$\liminf_{n \rightarrow \infty} \min(x_r - \lambda_{\max}(\mathbf{B}_n), \lambda_{\min}(\mathbf{B}_n) - x_l) > 0.$$

Since  $F^{\mathbf{B}_n} \rightarrow F^{c, H_1, H_2}$  with probability 1 the support of  $F^{c, H_1, H_2}$  is contained in interval (2.1) with probability 1. Thus, by (2.7), for  $f \in \{f_1, \dots, f_k\}$  and large  $n$ , with probability 1,

$$\int f(x) dG_n(x) = -\frac{1}{2\pi i} \oint f(z) M_n(z) dz$$

where the complex integral is over  $C \cup \overline{C}$ . For  $v \in [0, n^{-1} \varepsilon_n]$ , note that

$$|M_n(x_r + iv) - M_n(x_r + in^{-1} \varepsilon_n)| \leq 4n |\max(\lambda_{\max}(\mathbf{B}_n), e_r) - x_r|^{-1}$$

and

$$|M_n(x_l + iv) - M_n(x_l + in^{-1} \varepsilon_n)| \leq 4n |\min(\lambda_{\min}(\mathbf{B}_n), e_l) - x_l|^{-1}.$$

It follows that for large  $n$ , with probability 1,

$$\begin{aligned} & \left| \oint f(z) (M_n(z) - \widehat{M}_n(z)) dz \right| \\ & \leq 8K \varepsilon_n \left[ |\max(\lambda_{\max}(\mathbf{B}_n), e_r) - x_r|^{-1} + |\min(\lambda_{\min}(\mathbf{B}_n), e_l) - x_l|^{-1} \right] \rightarrow 0 \end{aligned}$$

where  $e_l$  ( $e_r$ ) is the left endpoint (right endpoint) of interval (2.1) and  $K$  is the bound on  $f$  over  $C$ .

Note that the mapping

$$\widehat{M}_n(\cdot) \rightarrow \left( -\frac{1}{2\pi i} \oint f_1(z) \widehat{M}_n(z) dz, \dots, -\frac{1}{2\pi i} \oint f_k(z) \widehat{M}_n(z) dz \right)$$

is continuous. Using Lemma 2.3, we complete the proof of Theorem 2.1.  $\square$

## 3. THE GAUSSIAN CASE

This section is to prove Lemma 2.3 under the Gaussian case, i.e.,  $\{x_{jk}\}$ ,  $j = 1, \dots, N, k = 1, \dots, n$  are standard normal random variables. Since  $\mathbf{T}_{1n}$  is symmetric there exists an orthogonal matrix  $\mathbf{U}$  such that

$$\mathbf{T}_{1n} = \mathbf{U} \text{diag}(s_1, \dots, s_n) \mathbf{U}'.$$

Note that  $\mathbf{X}_n$  has the same distribution as  $\mathbf{X}_n \mathbf{U}$ . It then suffices to consider

$$\widetilde{\mathbf{B}}_n = \frac{1}{N} \sum_{k=1}^n s_k \mathbf{T}_{2n}^{1/2} \mathbf{x}_k \mathbf{x}_k' \mathbf{T}_{2n}^{1/2} \triangleq \frac{1}{N} \sum_{k=1}^n s_k \mathbf{y}_k \mathbf{y}_k'$$

where  $\mathbf{x}_k$  is the  $k$ -th column of  $\mathbf{X}_n$ . In what follows, we omit the symbol  $\sim$  from the notation of  $\widetilde{\mathbf{B}}_n$  in order to simplify notation. Rewrite for  $z \in C_n$

$$M_n(z) = N[m_n(z) - Em_n(z)] + N[Em_n(z) - m_n^0(z)] \triangleq M_{n1}(z) + M_{n2}(z).$$

We below consider the random part  $M_{n1}(z)$  and the nonrandom part  $M_{n2}(z)$  separately to complete the proof of Lemma 2.3.

In the sequel we assume  $x_{jk}$ ,  $j = 1, \dots, N, k = 1, \dots, n$  are truncated at  $\delta_n \sqrt{n}$ , centralized and re-normalized. The details are omitted which is similar to Bai and Silverstein (2004).

We start with two probability inequalities for extreme eigenvalues of  $\mathbf{B}_n$ . It is well known (see [22],[1]) that for any  $l$ ,  $\eta_1 > (1 + \sqrt{c})^2$  and  $\eta_2 < (1 - \sqrt{c})^2$

$$\mathbf{P}\left(\lambda_{\max}\left(\frac{1}{N} \mathbf{X}_n' \mathbf{X}_n\right) \geq \eta_1\right) = o(n^{-l})$$

and

$$\mathbf{P}\left(\lambda_{\min}\left(\frac{1}{N} \mathbf{X}_n' \mathbf{X}_n\right) \leq \eta_2\right) = o(n^{-l}).$$

Thus, letting

$$\eta_r \in \begin{cases} (0, x_r), & c \geq 1, \\ (\limsup_n s_1 \lambda_{\max}^{\mathbf{T}_{2n}} (1 + \sqrt{c})^2, x_r), & \text{otherwise,} \end{cases}$$

we have for any  $l > 0$

$$(3.1) \quad \mathbf{P}(\lambda_{\max}(\mathbf{B}_n) \geq \eta_r) = o(n^{-l}).$$

Likewise, we have

$$(3.2) \quad \mathbf{P}(\lambda_{\min}(\mathbf{B}_n) \leq \eta_l) = o(n^{-l}).$$

where

$$\eta_l \in \begin{cases} (x_l, 0), & c \geq 1, \\ (x_l, \liminf_n s_n \lambda_{\min}^{\mathbf{T}_{2n}} I_{(0,1)}(c) (1 - \sqrt{c})^2), & \text{if } \liminf_n s_n \lambda_{\min}^{\mathbf{T}_{2n}} I_{(0,1)}(c) > 0, \\ (x_l, \liminf_n s_n \lambda_{\max}^{\mathbf{T}_{2n}} (1 + \sqrt{c})^2), & \text{if } \liminf_n s_n \lambda_{\min}^{\mathbf{T}_{2n}} I_{(0,1)}(c) \leq 0. \end{cases}$$

Here  $\eta_l, \eta_r, x_l, x_r$  can be chosen such that

$$(3.3) \quad x_r - \eta_r > 2\tau^2 \quad \text{and} \quad \eta_l - x_l > 2\tau^2,$$

where  $\tau$  are the upper bound of the spectral norms of  $\mathbf{T}_{1n}$  and  $\mathbf{T}_{2n}$  defined before.

**3.1. The limiting distribution of  $M_{n1}(z)$ .** The aim of this part is to find the limiting distribution of  $M_{n1}(z)$ . That is to say, we show for any positive integer  $r$ , the sum

$$\sum_{j=1}^r \alpha_j M_{n1}(z_j) \quad \Im z_j \neq 0$$

converges in distribution to a Gaussian random variable. We will use the central limit theorem for martingale difference sequences to accomplish the goal. Since

$$\lim_{v_0 \downarrow 0} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \int_{C_l \cup C_r} f(z) M_{n1}(z) dz \right|^2 \rightarrow 0,$$

it suffices to consider  $z = u + iv_0 \in C_u$ . Introduce

$$(3.4) \quad \begin{aligned} \mathbf{D}(z) &= \mathbf{B}_n(z) - z\mathbf{I}_N, \quad \mathbf{D}_k(z) = \mathbf{D}(z) - \frac{1}{N} s_k \mathbf{y}_k \mathbf{y}_k', \\ \mathbf{B}_{nk} &= \mathbf{B}_n - \frac{1}{N} s_k \mathbf{y}_k \mathbf{y}_k', \quad g_{2n}(z) = \frac{1}{N} \text{tr}(\mathbf{D}^{-1}(z) \mathbf{T}_{2n}), \end{aligned}$$

and

$$(3.5) \quad \begin{aligned} \varepsilon_k(z) &= \mathbf{y}_k' \mathbf{D}_k^{-1}(z) \mathbf{y}_k - \text{tr}(\mathbf{D}_k^{-1}(z) \mathbf{T}_{2n}), \quad \gamma_k(z) = \mathbf{y}_k' \mathbf{D}_k^{-2}(z) \mathbf{y}_k - \text{tr}(\mathbf{D}_k^{-2}(z) \mathbf{T}_{2n}) \\ \beta_k(z) &= \frac{1}{1 + N^{-1} s_k \mathbf{y}_k' \mathbf{D}_k^{-1}(z) \mathbf{y}_k}, \quad \tilde{\beta}_k(z) = \frac{1}{1 + N^{-1} s_k \text{tr}(\mathbf{D}_k^{-1}(z) \mathbf{T}_{2n})}, \end{aligned}$$

$$(3.6) \quad b_k(z) = \frac{1}{1 + N^{-1} s_k \mathbb{E} \text{tr}(\mathbf{D}_k^{-1}(z) \mathbf{T}_{2n})}, \quad \psi_k(z) = \frac{1}{1 + s_k \mathbb{E} g_{2n}(z)}.$$

Note that

$$m_n(z) = \frac{1}{N} \text{tr}(\mathbf{B}_n(z) - z\mathbf{I}_N)^{-1} \triangleq \frac{1}{N} \text{tr} \mathbf{D}^{-1}(z).$$

Let  $\mathbb{E}_0(\cdot)$  denote mathematical expectation and  $\mathbb{E}_k(\cdot)$  denote conditional expectation with respect to the  $\sigma$ -field given by  $\mathbf{x}_1, \dots, \mathbf{x}_k$ . By the formula

$$(3.7) \quad (\boldsymbol{\Sigma} + q\boldsymbol{\alpha}\boldsymbol{\beta}')^{-1} = \boldsymbol{\Sigma}^{-1} - \frac{q\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}\boldsymbol{\beta}'\boldsymbol{\Sigma}^{-1}}{1 + q\boldsymbol{\beta}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}},$$

we have

$$\begin{aligned}
M_{n1}(z) &= \sum_{k=1}^n \text{tr} \left\{ \mathbf{E}_k \mathbf{D}^{-1}(z) - \mathbf{E}_{k-1} \mathbf{D}^{-1}(z) \right\} = \sum_{k=1}^n (\mathbf{E}_k - \mathbf{E}_{k-1}) \text{tr} \left[ \mathbf{D}(z)^{-1} - \mathbf{D}_k^{-1}(z) \right] \\
&= -\frac{1}{N} \sum_{k=1}^n (\mathbf{E}_k - \mathbf{E}_{k-1}) s_k \beta_k(z) \mathbf{y}_k' \mathbf{D}_k^{-2}(z) \mathbf{y}_k \\
&= -\frac{1}{N} \sum_{k=1}^n (\mathbf{E}_k - \mathbf{E}_{k-1}) s_k \beta_k(z) \gamma_k(z) - \frac{1}{N} \sum_{k=1}^n (\mathbf{E}_k - \mathbf{E}_{k-1}) s_k \beta_k(z) \text{tr}(\mathbf{D}_k^{-2}(z) \mathbf{T}_{2n}) \\
(3.8) \quad &\triangleq \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}$$

From the identity

$$(3.9) \quad \beta_k(z) - \widetilde{\beta}_k(z) = -\frac{1}{N} s_k \widetilde{\beta}_k(z) \beta_k(z) \varepsilon_k(z),$$

we have

$$\mathcal{I}_1 = -\frac{1}{N} \sum_{k=1}^n \mathbf{E}_k s_k \widetilde{\beta}_k(z) \gamma_k(z) + \frac{1}{N^2} \sum_{k=1}^n (\mathbf{E}_k - \mathbf{E}_{k-1}) s_k^2 \widetilde{\beta}_k(z) \beta_k(z) \varepsilon_k(z) \gamma_k(z).$$

By Lemma A.1 and Lemma B.1

$$\begin{aligned}
&\frac{1}{N^4} \sum_{k=1}^n \mathbf{E} |(\mathbf{E}_k - \mathbf{E}_{k-1}) s_k^2 \widetilde{\beta}_k(z) \beta_k(z) \varepsilon_k(z) \gamma_k(z)|^2 \\
&\leq \frac{C}{N^4} \sum_{k=1}^n \mathbf{E}^{1/2} |\widetilde{\beta}_k(z) \beta_k(z)|^2 \mathbf{E}^{1/2} |\varepsilon_k(z) \gamma_k(z)|^4 \\
&\leq \frac{C}{N^4} \sum_{k=1}^n \mathbf{E}^{1/4} |\varepsilon_k(z)|^8 \mathbf{E}^{1/4} |\gamma_k(z)|^8 \leq \frac{C}{N} \rightarrow 0.
\end{aligned}$$

This implies

$$(3.10) \quad \mathcal{I}_1 = -\frac{1}{N} \sum_{k=1}^n \mathbf{E}_k s_k \widetilde{\beta}_k(z) \gamma_k(z) + o_p(1).$$

Using the same argument and

$$(3.11) \quad \beta_k(z) - \widetilde{\beta}_k(z) = -\frac{1}{N} s_k \widetilde{\beta}_k^2(z) \varepsilon_k(z) + \frac{1}{N^2} s_k^2 \beta_k(z) \widetilde{\beta}_k^2(z) \varepsilon_k^2(z),$$

one gets

$$(3.12) \quad \mathcal{I}_2 = \frac{1}{N^2} \sum_{k=1}^n \mathbf{E}_k s_k^2 \widetilde{\beta}_k^2(z) \varepsilon_k(z) \text{tr}(\mathbf{D}_k^{-2}(z) \mathbf{T}_{2n}) + o_p(1).$$

From (3.8), (3.10), and (3.12), we conclude that

$$M_{n1}(z) = -\frac{1}{N} \sum_{k=1}^n \mathbb{E}_k s_k \widetilde{\beta}_k(z) \gamma_k(z) + \frac{1}{N^2} \sum_{k=1}^n \mathbb{E}_k s_k^2 \widetilde{\beta}_k^2(z) \varepsilon_k(z) \text{tr}(\mathbf{D}_k^{-2}(z) \mathbf{T}_{2n}) + o_p(1).$$

Define

$$\begin{aligned} h_k(z) &= -\frac{1}{N} \mathbb{E}_k s_k \widetilde{\beta}_k(z) \gamma_k(z) + \frac{1}{N^2} \mathbb{E}_k s_k^2 \widetilde{\beta}_k^2(z) \varepsilon_k(z) \text{tr}(\mathbf{D}_k^{-2}(z) \mathbf{T}_{2n}) \\ &= -N^{-1} \frac{d}{dz} \mathbb{E}_k s_k \widetilde{\beta}_k(z) \varepsilon_k(z). \end{aligned}$$

Thus we only need to prove that  $\sum_{j=1}^r \alpha_j \sum_{k=1}^n h_k(z_j) = \sum_{k=1}^n \sum_{j=1}^r \alpha_j h_k(z_j)$  converges in distribution to a Gaussian random variable. By Lemma B.2, it suffices to verify condition (i) and (ii). It follows from Lemma A.1 and Lemma B.1 that

$$\begin{aligned} \sum_{k=1}^n \mathbb{E} \left| \sum_{j=1}^r \alpha_j h_k(z_j) \right|^4 &\leq \frac{C}{N^4} \sum_{k=1}^n \sum_{j=1}^r \alpha_j^4 \left[ \mathbb{E}^{1/2} |\widetilde{\beta}_k|^8 \mathbb{E}^{1/2} |\gamma_k(z_j)|^8 \right. \\ &\quad \left. + \mathbb{E}^{1/2} |\widetilde{\beta}_k|^{16} \mathbb{E}^{1/2} |\varepsilon_k(z_j)|^8 \right] \leq \frac{C}{N} \rightarrow 0 \end{aligned}$$

which implies that conditions (ii) of Lemma B.2 is satisfied.

The next aim is to find a limit in probability of

$$\Phi(z_1, z_2) \triangleq \sum_{k=1}^n \mathbb{E}_{k-1} [h_k(z_1) h_k(z_2)]$$

for  $z_1, z_2$  with nonzero fixed imaginary parts. It is obvious that

$$\Phi(z_1, z_2) = N^{-2} \frac{\partial^2}{\partial z_2 \partial z_1} \sum_{k=1}^n \mathbb{E}_{k-1} \left[ \mathbb{E}_k \left( s_k \widetilde{\beta}_k(z_1) \varepsilon_k(z_1) \right) \mathbb{E}_k \left( s_k \widetilde{\beta}_k(z_2) \varepsilon_k(z_2) \right) \right].$$

Due to the analysis on page 571 in [1], it is enough to prove that

$$N^{-2} \sum_{k=1}^n s_k^2 \mathbb{E}_{k-1} \left[ \mathbb{E}_k \left( \widetilde{\beta}_k(z_1) \varepsilon_k(z_1) \right) \mathbb{E}_k \left( \widetilde{\beta}_k(z_2) \varepsilon_k(z_2) \right) \right]$$

converges in probability to a constant. Similar to (A.3) in the appendix, it can be verified that  $|\widetilde{\beta}_k(z)|$  and  $|b_k(z)|$  has the same bound as  $\beta_k(z)$ . Combining (3.1), (3.2), with (3.7), we have for  $l = 1, 2$  and suitably large  $t$

(3.13)

$$\begin{aligned} &\mathbb{E} |\widetilde{\beta}_k(z_l) - b_k(z_l)|^2 \\ &\leq \mathbb{E} |\widetilde{\beta}_k(z_l) - b_k(z_l)|^2 I(\eta_l \leq \lambda_{\min} \leq \lambda_{\max} \leq \eta_r) + \mathbb{E} |\widetilde{\beta}_k(z_l) - b_k(z_l)|^2 \\ &\quad \times I(\lambda_{\min} < \eta_l \text{ or } \lambda_{\max} > \eta_r) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{C}{N^2} \mathbb{E} |\text{tr}(\mathbf{D}_k^{-1}(z_l) \mathbf{T}_{2n}) - \mathbb{E} \text{tr}(\mathbf{D}_k^{-1}(z_l) \mathbf{T}_{2n})|^2 \\
&\quad + CNE |\text{tr}(\mathbf{D}_k^{-1}(z_l) \mathbf{T}_{2n}) - \mathbb{E} \text{tr}(\mathbf{D}_k^{-1}(z_l) \mathbf{T}_{2n})|^2 I(\lambda_{\min} < \eta_l \text{ or } \lambda_{\max} > \eta_r) \\
&\leq \frac{C}{N^2} \mathbb{E} \left| \sum_{j=1, j \neq k}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) \left[ \text{tr}(\mathbf{D}_k^{-1}(z_l)) - \text{tr}(\mathbf{D}_{kj}^{-1}(z_l)) \right] \right|^2 \\
&\quad + CNE \left| \sum_{j=1, j \neq k}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) \left[ \text{tr}(\mathbf{D}_k^{-1}(z_l)) - \text{tr}(\mathbf{D}_{kj}^{-1}(z_l)) \right] \right|^2 I(\lambda_{\min} < \eta_l \text{ or } \lambda_{\max} > \eta_r) \\
&\leq \frac{C}{N^2} \sum_{j=1, j \neq k}^n \mathbb{E} |\text{tr}(\mathbf{D}_k^{-1}(z_l)) - \text{tr}(\mathbf{D}_{kj}^{-1}(z_l))|^2 \\
&\quad + CN^2 \sum_{j=1, j \neq k}^n \mathbb{E} |\text{tr}(\mathbf{D}_k^{-1}(z_l)) - \text{tr}(\mathbf{D}_{kj}^{-1}(z_l))|^2 I(\lambda_{\min} < \eta_l \text{ or } \lambda_{\max} > \eta_r) \\
&= \frac{C}{N^2} \sum_{j=1, j \neq k}^n \mathbb{E} \left| \frac{s_j \mathbf{y}_j' \mathbf{D}_{kj}^{-2}(z_l) \mathbf{y}_j}{N(1 + N^{-1} s_j \mathbf{y}_j' \mathbf{D}_{kj}^{-1}(z_l) \mathbf{y}_j)} \right|^2 \\
&\quad + CN^2 \sum_{j=1, j \neq k}^n \mathbb{E} \left| \frac{s_j \mathbf{y}_j' \mathbf{D}_{kj}^{-2}(z_l) \mathbf{y}_j}{N(1 + N^{-1} s_j \mathbf{y}_j' \mathbf{D}_{kj}^{-1}(z_l) \mathbf{y}_j)} \right|^2 I(\lambda_{\min} < \eta_l \text{ or } \lambda_{\max} > \eta_r) \\
&\leq \frac{C}{N} + CN^3 \mathbb{P}(\lambda_{\min} < \eta_l \text{ or } \lambda_{\max} > \eta_r) \leq \frac{C}{N} + CN^3 n^{-t} \rightarrow 0
\end{aligned}$$

where  $\mathbf{D}_{kj}(z) = \mathbf{D}_j(z) - \frac{1}{N} s_j \mathbf{y}_j \mathbf{y}_j'$ . From the above inequality, we get

$$\begin{aligned}
&\mathbb{E} \left| N^{-2} \sum_{k=1}^n s_k^2 \mathbb{E}_{k-1} \left[ \mathbb{E}_k \left( \widetilde{\beta}_k(z_1) \varepsilon_k(z_1) \right) \mathbb{E}_k \left( \widetilde{\beta}_k(z_2) \varepsilon_k(z_2) \right) \right. \right. \\
&\quad \left. \left. - \mathbb{E}_k \left( b_k(z_1) \varepsilon_k(z_1) \right) \mathbb{E}_k \left( b_k(z_2) \varepsilon_k(z_2) \right) \right] \right| \\
&\leq CN^{-2} \sum_{k=1}^n \left[ \mathbb{E} |\mathbb{E}_k \left( \widetilde{\beta}_k(z_1) - b_k(z_1) \right) \varepsilon_k(z_1)| \mathbb{E}_k \left( \widetilde{\beta}_k(z_2) \varepsilon_k(z_2) \right)| \right. \\
&\quad \left. + \mathbb{E} |\mathbb{E}_k \left( b_k(z_1) \varepsilon_k(z_1) \right) \mathbb{E}_k \left( \widetilde{\beta}_k(z_2) - b_k(z_2) \right) \varepsilon_k(z_2)| \right] \\
&\leq CN^{-2} \sum_{k=1}^n \left[ \mathbb{E}^{1/2} |\widetilde{\beta}_k(z_1) - b_k(z_1)|^2 \mathbb{E}^{1/2} |\widetilde{\beta}_k(z_2) \varepsilon_k(z_2)|^2 \right. \\
&\quad \left. + \mathbb{E}^{1/2} |b_k(z_1) \varepsilon_k(z_1)|^2 \mathbb{E}^{1/2} |(\widetilde{\beta}_k(z_2) - b_k(z_2)) \varepsilon_k(z_2)|^2 \right] \\
&\leq CN^{-1} \sum_{k=1}^n \left[ \mathbb{E}^{1/2} |\widetilde{\beta}_k(z_1) - b_k(z_1)|^2 + \mathbb{E}^{1/2} |(\widetilde{\beta}_k(z_2) - b_k(z_2))|^2 \right] \rightarrow 0,
\end{aligned}$$

which yields

$$N^{-2} \sum_{k=1}^n s_k^2 \mathbf{E}_{k-1} \left[ \mathbf{E}_k \left( \widetilde{\beta}_k(z_1) \varepsilon_k(z_1) \right) \mathbf{E}_k \left( \widetilde{\beta}_k(z_2) \varepsilon_k(z_2) \right) - \mathbf{E}_k \left( b_k(z_1) \varepsilon_k(z_1) \right) \mathbf{E}_k \left( b_k(z_2) \varepsilon_k(z_2) \right) \right] \xrightarrow{i.p.} 0.$$

Therefore, our goal is to find the limit in probability of

$$N^{-2} \sum_{k=1}^n s_k^2 b_k(z_1) b_k(z_2) \mathbf{E}_{k-1} [\mathbf{E}_k (\varepsilon_k(z_1)) \mathbf{E}_k (\varepsilon_k(z_2))].$$

Using the moments of normal random variables, we have

$$\mathbf{E}_{k-1} [\mathbf{E}_k (\varepsilon_k(z_1)) \mathbf{E}_k (\varepsilon_k(z_2))] = 2\text{tr}(\mathbf{T}_{2n} \mathbf{E}_k \mathbf{D}_k^{-1}(z_1) \mathbf{T}_{2n} \mathbf{E}_k \mathbf{D}_k^{-1}(z_2)).$$

Consequently, it suffices to study

$$(3.14) \quad N^{-2} \sum_{k=1}^n s_k^2 b_k(z_1) b_k(z_2) \text{tr}(\mathbf{T}_{2n} \mathbf{E}_k \mathbf{D}_k^{-1}(z_1) \mathbf{T}_{2n} \mathbf{E}_k \mathbf{D}_k^{-1}(z_2)).$$

$$\text{Let } \mathbf{R}_k(z) = z\mathbf{I} - \frac{1}{N} \sum_{j \neq k} s_j \psi_j(z) \mathbf{T}_{2n},$$

$$\beta_{jk}(z) = \frac{1}{1 + N^{-1} s_j \mathbf{y}_j' \mathbf{D}_{jk}^{-1}(z) \mathbf{y}_j} \quad \text{and} \quad b_{jk}(z) = \frac{1}{1 + N^{-1} s_j \text{Etr}(\mathbf{D}_{jk}^{-1}(z) \mathbf{T}_{2n})}.$$

Write

$$\mathbf{D}_k(z_1) + \mathbf{R}_k(z_1) = \frac{1}{N} \sum_{j \neq k} s_j \mathbf{y}_j \mathbf{y}_j' - \frac{1}{N} \sum_{j \neq k} s_j \psi_j(z_1) \mathbf{T}_{2n}$$

which implies that

$$\mathbf{R}_k^{-1}(z_1) + \mathbf{D}_k^{-1}(z_1) = \frac{1}{N} \sum_{j \neq k} s_j \mathbf{R}_k^{-1}(z_1) \mathbf{y}_j \mathbf{y}_j' \mathbf{D}_k^{-1}(z_1) - \frac{1}{N} \sum_{j \neq k} s_j \psi_j(z_1) \mathbf{R}_k^{-1}(z_1) \mathbf{T}_{2n} \mathbf{D}_k^{-1}(z_1).$$

Using the formula

$$(3.15) \quad (\mathbf{\Sigma} + q\alpha\beta')^{-1} \alpha = \frac{\mathbf{\Sigma}^{-1} \alpha}{1 + q\beta' \mathbf{\Sigma}^{-1} \alpha},$$

we have

$$(3.16) \quad \begin{aligned} \mathbf{R}_k^{-1}(z_1) + \mathbf{D}_k^{-1}(z_1) &= \frac{1}{N} \sum_{j \neq k} s_j \psi_j(z_1) \mathbf{R}_k^{-1}(z_1) (\mathbf{y}_j \mathbf{y}_j' - \mathbf{T}_{2n}) \mathbf{D}_{jk}^{-1}(z_1) \\ &\quad + \frac{1}{N} \sum_{j \neq k} s_j (\beta_{jk}(z_1) - \psi_j(z_1)) \mathbf{R}_k^{-1}(z_1) \mathbf{y}_j \mathbf{y}_j' \mathbf{D}_{jk}^{-1}(z_1) \\ &\quad + \frac{1}{N} \sum_{j \neq k} s_j \psi_j(z_1) \mathbf{R}_k^{-1}(z_1) \mathbf{T}_{2n} (\mathbf{D}_{jk}^{-1}(z_1) - \mathbf{D}_k^{-1}(z_1)) \end{aligned}$$



$$\triangleq \mathbf{A}_1(z_1) + \mathbf{A}_2(z_1) + \mathbf{A}_3(z_1).$$

By a direct calculation, we have for any positive number  $t \geq 0$

$$\begin{aligned} \Im \left( z - \frac{1}{N} \sum_{j \neq k} s_j \psi_j(z) t \right) &= v_0 - \frac{1}{N} \sum_{j \neq k} \frac{s_j^2 t}{|1 + s_j \mathbf{E} g_{2n}(z)|^2} \Im \mathbf{E} g_{2n}(\bar{z}) \\ &= v_0 \left( 1 + \frac{1}{N^2} \sum_{j \neq k} \frac{s_j^2 t}{|1 + s_j \mathbf{E} g_{2n}(z)|^2} \mathbf{E} \text{tr} \left( \mathbf{D}^{-1}(z) \mathbf{D}^{-1}(\bar{z}) \mathbf{T}_{2n} \right) \right) \geq v_0 \end{aligned}$$

which yields

$$\|\mathbf{R}_k^{-1}(z)\| \leq \frac{1}{v_0}.$$

Let  $\mathbf{M}$  be a  $N \times N$  matrix with a nonrandom bound on the spectral norm of  $\mathbf{M}$  for all parameters governing  $\mathbf{M}$  and under all realizations of  $\mathbf{M}$ . By the Cauchy-Schwarz inequality, one gets

$$\begin{aligned} (3.17) \quad \mathbf{E} \left| \text{tr}(\mathbf{A}_1(z_1) \mathbf{M}) \right| &\leq C \mathbf{E}^{1/2} \left| \mathbf{y}_j' \mathbf{D}_{jk}^{-1}(z_1) \mathbf{R}_k^{-1}(z_1) \mathbf{y}_j \right. \\ &\quad \left. - \text{tr}(\mathbf{R}_k^{-1}(z_1) \mathbf{T}_{2n} \mathbf{D}_{jk}^{-1}(z_1)) \right|^2 = O(N^{1/2}). \end{aligned}$$

Let  $\widetilde{\beta}_{jk}(z) = \frac{1}{1 + N^{-1} s_j \text{tr}(\mathbf{D}_{jk}^{-1}(z) \mathbf{T}_{2n})}$ . From (3.13)

$$\mathbf{E} |\widetilde{\beta}_{jk}(z) - b_{jk}(z)| = O(N^{-1}).$$

Applying the above inequality, Lemma B.1 and Lemma B.3, we obtain

$$\begin{aligned} (3.18) \quad \mathbf{E} |\beta_{jk}(z) - \psi_j(z)|^2 &\leq C \left[ \mathbf{E} |\beta_{jk}(z) - \widetilde{\beta}_{jk}(z)|^2 + |b_{jk}(z) - \psi_j(z)|^2 \right] + O(N^{-1}) \\ &\leq \frac{C}{N^2} \mathbf{E}^{1/2} |\mathbf{y}_j' \mathbf{D}_{jk}^{-1}(z) \mathbf{y}_j - \text{tr}(\mathbf{D}_{jk}^{-1}(z) \mathbf{T}_{2n})|^4 \\ &\quad + \frac{C}{N^2} \mathbf{E} |\text{tr}(\mathbf{D}_{jk}^{-1}(z) \mathbf{T}_{2n}) - \text{tr}(\mathbf{D}^{-1}(z) \mathbf{T}_{2n})|^2 + O(N^{-1}) \\ &\leq \frac{C}{N} + \frac{C}{N^2} + O(N^{-1}) = O(N^{-1}) \end{aligned}$$

which implies that

$$\begin{aligned} (3.19) \quad \mathbf{E} \left| \text{tr}(\mathbf{A}_2(z_1) \mathbf{M}) \right| &\leq \frac{C}{N} \sum_{j \neq k} \mathbf{E}^{1/2} |\beta_{jk}(z_1) - \psi_j(z_1)|^2 \\ &\quad \times \mathbf{E}^{1/2} |\mathbf{y}_j' \mathbf{D}_{jk}^{-1}(z_1) \mathbf{M} \mathbf{R}_k^{-1}(z_1) \mathbf{y}_j|^2 = O(N^{1/2}). \end{aligned}$$

Lemma B.3 implies that

$$(3.20) \quad \mathbb{E} |\text{tr}(\mathbf{A}_3(z_1)\mathbf{M})| \leq \frac{C}{N} \sum_{j \neq k} \text{tr} \left[ \left( \mathbf{D}_{jk}^{-1}(z_1) - \mathbf{D}_k^{-1}(z_1) \right) \times \mathbf{R}_k^{-1}(z_1) \mathbf{T}_{2n} \right] \leq C.$$

Using (3.16), (3.19), and (3.20), one gets

$$(3.21) \quad \begin{aligned} \text{tr}(\mathbf{E}_k(\mathbf{D}_k(z_1)) \mathbf{T}_{2n} \mathbf{D}_k^{-1}(z_2) \mathbf{T}_{2n}) &= -\text{tr}(\mathbf{E}_k(\mathbf{R}_k^{-1}(z_1)) \mathbf{T}_{2n} \mathbf{D}_k^{-1}(z_2) \mathbf{T}_{2n}) \\ &\quad + \text{tr}(\mathbf{E}_k(\mathbf{A}_1(z_1)) \mathbf{T}_{2n} \mathbf{D}_k^{-1}(z_2) \mathbf{T}_{2n}) + a(z_1, z_2) \end{aligned}$$

where  $\mathbb{E}|a(z_1, z_2)| \leq O(N^{1/2})$ . Furthermore, write

$$\begin{aligned} &\text{tr}(\mathbf{E}_k(\mathbf{A}_1(z_1)) \mathbf{T}_{2n} \mathbf{D}_k^{-1}(z_2) \mathbf{T}_{2n}) \\ &= -\frac{1}{N^2} \sum_{j < k} s_j^2 \psi_j(z_1) \beta_{jk}(z_2) \left[ \mathbf{y}_j' \mathbf{E}_k \mathbf{D}_{jk}^{-1}(z_1) \mathbf{T}_{2n} \mathbf{D}_{jk}^{-1}(z_2) \mathbf{y}_j - \text{tr}(\mathbf{E}_k \mathbf{D}_{jk}^{-1}(z_1) \mathbf{T}_{2n} \mathbf{D}_{jk}^{-1}(z_2) \mathbf{T}_{2n}) \right] \\ &\quad \times \left[ \mathbf{y}_j' \mathbf{D}_{jk}^{-1}(z_2) \mathbf{T}_{2n} \mathbf{R}_k^{-1}(z_1) \mathbf{y}_j - \text{tr}(\mathbf{D}_{jk}^{-1}(z_2) \mathbf{T}_{2n} \mathbf{R}_k^{-1}(z_1) \mathbf{T}_{2n}) \right] \\ &\quad - \frac{1}{N^2} \sum_{j < k} s_j^2 \psi_j(z_1) \beta_{jk}(z_2) \left( \mathbf{y}_j' \mathbf{E}_k \mathbf{D}_{jk}^{-1}(z_1) \mathbf{T}_{2n} \mathbf{D}_{jk}^{-1}(z_2) \mathbf{y}_j - \text{tr}(\mathbf{E}_k \mathbf{D}_{jk}^{-1}(z_1) \mathbf{T}_{2n} \mathbf{D}_{jk}^{-1}(z_2) \mathbf{T}_{2n}) \right) \\ &\quad \times \text{tr}(\mathbf{D}_{jk}^{-1}(z_2) \mathbf{T}_{2n} \mathbf{R}_k^{-1}(z_1) \mathbf{T}_{2n}) \\ &\quad - \frac{1}{N^2} \sum_{j < k} s_j^2 \psi_j(z_1) \beta_{jk}(z_2) \text{tr}(\mathbf{E}_k \mathbf{D}_{jk}^{-1}(z_1) \mathbf{T}_{2n} \mathbf{D}_{jk}^{-1}(z_2) \mathbf{T}_{2n}) \\ &\quad \times \left[ \mathbf{y}_j' \mathbf{D}_{jk}^{-1}(z_2) \mathbf{T}_{2n} \mathbf{R}_k^{-1}(z_1) \mathbf{y}_j - \text{tr}(\mathbf{D}_{jk}^{-1}(z_2) \mathbf{T}_{2n} \mathbf{R}_k^{-1}(z_1) \mathbf{T}_{2n}) \right] \\ &\quad - \frac{1}{N^2} \sum_{j < k} s_j^2 \psi_j(z_1) \beta_{jk}(z_2) \text{tr}(\mathbf{E}_k \mathbf{D}_{jk}^{-1}(z_1) \mathbf{T}_{2n} \mathbf{D}_{jk}^{-1}(z_2) \mathbf{T}_{2n}) \text{tr}(\mathbf{D}_{jk}^{-1}(z_2) \mathbf{T}_{2n} \mathbf{R}_k^{-1}(z_1) \mathbf{T}_{2n}) \\ &\quad + \frac{1}{N} \sum_{j < k} s_j \psi_j(z_1) \text{tr} \left[ \mathbf{R}_k^{-1}(z_1) (\mathbf{y}_j \mathbf{y}_j' - \mathbf{T}_{2n}) \mathbf{E}_k \mathbf{D}_{jk}^{-1}(z_1) \mathbf{T}_{2n} \mathbf{D}_{jk}^{-1}(z_2) \mathbf{T}_{2n} \right] \\ &\quad - \frac{1}{N} \sum_{j < k} s_j \psi_j(z_1) \text{tr} \left[ \mathbf{R}_k^{-1}(z_1) \mathbf{T}_{2n} \mathbf{E}_k \mathbf{D}_{jk}^{-1}(z_1) \mathbf{T}_{2n} (\mathbf{D}_k^{-1}(z_2) - \mathbf{D}_{jk}^{-1}(z_2)) \mathbf{T}_{2n} \right] \\ &\triangleq a_1(z_1, z_2) + a_2(z_1, z_2) + a_3(z_1, z_2) + a_4(z_1, z_2) + a_5(z_1, z_2) + a_6(z_1, z_2). \end{aligned}$$

It follows from Lemma A.1 and Lemma B.1 that

$$\mathbb{E}|a_1(z_1, z_2) + a_2(z_1, z_2) + a_3(z_1, z_2) + a_5(z_1, z_2)| \leq CN^{1/2}.$$

In addition, Lemma B.3 yields that

$$\mathbb{E}|a_6(z_1, z_2)| \leq C$$

and that

$$\mathbb{E}|a_4(z_1, z_2)| + \frac{1}{N^2} \sum_{j < k} s_j^2 \psi_j(z_1) \psi_j(z_2)$$

$$\operatorname{tr}(\mathbf{E}_k \mathbf{D}_k^{-1}(z_1) \mathbf{T}_{2n} \mathbf{D}_k^{-1}(z_2) \mathbf{T}_{2n}) \operatorname{tr}(\mathbf{D}_k^{-1}(z_2) \mathbf{T}_{2n} \mathbf{R}_k^{-1}(z_1) \mathbf{T}_{2n}) \leq CN^{-1}$$

where the last inequality uses (3.18). (3.21), together with the above three inequalities, ensures that

$$\begin{aligned} & \operatorname{tr}(\mathbf{E}_k \mathbf{D}_k^{-1}(z_1) \mathbf{T}_{2n} \mathbf{D}_k^{-1}(z_2) \mathbf{T}_{2n}) \left[ 1 + \frac{1}{N^2} \sum_{j < k} s_j^2 \psi_j(z_1) \psi_j(z_2) \operatorname{tr}(\mathbf{D}_k^{-1}(z_2) \mathbf{T}_{2n} \mathbf{R}_k^{-1}(z_1) \mathbf{T}_{2n}) \right] \\ &= -\operatorname{tr}(\mathbf{R}_k^{-1}(z_1) \mathbf{T}_{2n} \mathbf{D}_k^{-1}(z_2) \mathbf{T}_{2n}) + a_7(z_1, z_2) \end{aligned}$$

where  $\mathbb{E}|a_7(z_1, z_2)| \leq CN^{1/2}$ . Combining (3.17), (3.19) with (3.20), one has

$$\begin{aligned} (3.22) \quad & \operatorname{tr}(\mathbf{E}_k \mathbf{D}_k^{-1}(z_1) \mathbf{T}_{2n} \mathbf{D}_k^{-1}(z_2) \mathbf{T}_{2n}) \\ & \times \left[ 1 - \frac{1}{N^2} \sum_{j < k} s_j^2 \psi_j(z_1) \psi_j(z_2) \operatorname{tr}(\mathbf{R}_k^{-1}(z_1) \mathbf{T}_{2n} \mathbf{R}_k^{-1}(z_2) \mathbf{T}_{2n}) \right] \\ &= \operatorname{tr}(\mathbf{R}_k^{-1}(z_1) \mathbf{T}_{2n} \mathbf{R}_k^{-1}(z_2) \mathbf{T}_{2n}) + a_8(z_1, z_2) \end{aligned}$$

where  $\mathbb{E}|a_8(z_1, z_2)| \leq CN^{1/2}$ . From [23]

$$g_{2n}(z) \rightarrow g_2(z) \quad \text{a.s. as } n \rightarrow \infty.$$

It follows that

$$(3.23) \quad \left| \psi_j(z) - \frac{1}{1 + s_j g_{2n}^0(z)} \right| \leq C \left( |\mathbb{E} g_{2n}(z) - g_2(z)| + |g_{2n}^0(z) - g_2(z)| \right) = o(1),$$

where  $g_{2n}(z)$  is defined at (3.4). Note that by (1.9)

$$\begin{aligned} & \frac{1}{N} \sum_{j=1}^n s_j^2 \frac{1}{(1 + s_j g_{2n}^0(z_1))(1 + s_j g_{2n}^0(z_2))} \\ &= \frac{1}{g_{2n}^0(z_1) - g_{2n}^0(z_2)} \left[ \frac{1}{N} \sum_{j=1}^n \frac{s_j}{1 + s_j g_{2n}^0(z_2)} - \frac{1}{N} \sum_{j=1}^n \frac{s_j}{1 + s_j g_{2n}^0(z_1)} \right] \\ (3.24) \quad &= \frac{z_1 g_{1n}^0(z_1) - z_2 g_{1n}^0(z_2)}{g_{2n}^0(z_1) - g_{2n}^0(z_2)} \end{aligned}$$

and by (1.10)

$$(3.25) \quad \int \frac{t^2}{(1 + g_{1n}^0(z_1)t)(1 + g_{1n}^0(z_2)t)} dH_{2n}(t) = \frac{z_1 g_{2n}^0(z_1) - z_2 g_{2n}^0(z_2)}{g_{1n}^0(z_1) - g_{1n}^0(z_2)}.$$

Using the fact from (1.9) and (3.23) that

$$\frac{1}{N} \sum_{j \neq k} s_j \psi_j(z) + z g_{1n}^0(z) = o(1),$$

we deduce that

$$\begin{aligned} \text{tr}(\mathbf{R}_k^{-1}(z_1)\mathbf{T}_{2n}\mathbf{R}_k^{-1}(z_2)\mathbf{T}_{2n}) &= \frac{N}{z_1 z_2} \int \frac{t^2}{(1 + g_{1n}^0(z_1)t)(1 + g_{1n}^0(z_2)t)} dH_{2n}(t) \\ &= \frac{N}{z_1 z_2} \frac{z_1 g_{2n}^0(z_1) - z_2 g_{2n}^0(z_2)}{g_{1n}^0(z_1) - g_{1n}^0(z_2)}. \end{aligned}$$

We now deal with  $\frac{1}{N^2} \sum_{j < k} s_j^2 \psi_j(z_1) \psi_j(z_2)$  in (3.22). For any  $\varepsilon \in (0, 1/100)$ , we now distinguish the following two cases.

Case 1 : When  $k \leq n^{1-\varepsilon}$ , one gets

$$\begin{aligned} &\frac{1}{N^2} \sum_{j < k} \left| \left[ \frac{s_j^2}{(1 + s_j g_{2n}^0(z_1))(1 + s_j g_{2n}^0(z_2))} - c_n^{-1} \frac{z_1 g_{1n}^0(z_1) - z_2 g_{1n}^0(z_2)}{g_{2n}^0(z_1) - g_{2n}^0(z_2)} \right] \right. \\ &\quad \left. \times \text{tr}(\mathbf{R}_k^{-1}(z_1)\mathbf{T}_{2n}\mathbf{R}_k^{-1}(z_2)\mathbf{T}_{2n}) \right| \leq CN^{-\varepsilon} = o(1), \end{aligned}$$

Case 2 : When  $k > n^{1-\varepsilon}$ , one gets by (3.24)

$$\begin{aligned} &\frac{1}{N^2} \left| \sum_{j < k} \left[ \frac{s_j^2}{(1 + s_j g_{2n}^0(z_1))(1 + s_j g_{2n}^0(z_2))} - c_n^{-1} \frac{z_1 g_{1n}^0(z_1) - z_2 g_{1n}^0(z_2)}{g_{2n}^0(z_1) - g_{2n}^0(z_2)} \right] \right. \\ &\quad \left. \times \text{tr}(\mathbf{R}_k^{-1}(z_1)\mathbf{T}_{2n}\mathbf{R}_k^{-1}(z_2)\mathbf{T}_{2n}) \right| \\ &\leq \frac{1}{N} \left| \sum_{j < k} \left[ \frac{s_j^2}{(1 + s_j g_{2n}^0(z_1))(1 + s_j g_{2n}^0(z_2))} - c_n^{-1} \frac{z_1 g_{1n}^0(z_1) - z_2 g_{1n}^0(z_2)}{g_{2n}^0(z_1) - g_{2n}^0(z_2)} \right] \right| \\ &= o(1). \end{aligned}$$

It follows that

$$\begin{aligned} &\text{tr}(\mathbf{E}_k \mathbf{D}_k^{-1}(z_1)\mathbf{T}_{2n}\mathbf{D}_k^{-1}(z_2)\mathbf{T}_{2n}) \\ &\quad \times \left[ 1 - \frac{k-1}{nz_1 z_2} \frac{z_1 g_{1n}^0(z_1) - z_2 g_{1n}^0(z_2)}{g_{2n}^0(z_1) - g_{2n}^0(z_2)} \frac{z_1 g_{2n}^0(z_1) - z_2 g_{2n}^0(z_2)}{g_{1n}^0(z_1) - g_{1n}^0(z_2)} \right] \\ &= \frac{N}{z_1 z_2} \frac{z_1 g_{2n}^0(z_1) - z_2 g_{2n}^0(z_2)}{g_{1n}^0(z_1) - g_{1n}^0(z_2)} + a_9(z_1, z_2) \end{aligned}$$

where  $\mathbb{E}|a_9(z_1, z_2)| = o(N)$ . Applying Lemma B.3 and (3.23), one gets

$$\left| b_k(z) - \frac{1}{1 + s_k g_{2n}^0(z)} \right| \leq \frac{C}{N} \mathbb{E} \left| \text{tr}(\mathbf{D}_k^{-1}(z) - \mathbf{D}^{-1}(z)) \mathbf{T}_{2n} \right| + o(1) = o(1).$$

Set

$$f_n(z_1, z_2) = \frac{1}{z_1 z_2} \frac{z_1 g_{1n}^0(z_1) - z_2 g_{1n}^0(z_2)}{g_{2n}^0(z_1) - g_{2n}^0(z_2)} \frac{z_1 g_{2n}^0(z_1) - z_2 g_{2n}^0(z_2)}{g_{1n}^0(z_1) - g_{1n}^0(z_2)}$$

and

$$r_{nk}(z_1, z_2) = \frac{s_k^2}{(1 + s_k g_{2n}^0(z_1))(1 + s_k g_{2n}^0(z_2))}.$$

By (3.24) and (3.25), we obtain

(3.26)

$$\begin{aligned} |f_n(z_1, z_2)| &\leq \left[ c_n \int \frac{x^2}{|1 + g_{2n}^0(z_1)x|^2} dH_{1n}(x) \int \frac{t^2}{|z_1(1 + g_{1n}^0(z_1)t)|^2} dH_{2n}(t) \right]^{1/2} \\ &\quad \times \left[ c_n \int \frac{x^2}{|1 + g_{2n}^0(z_2)x|^2} dH_{1n}(x) \int \frac{t^2}{|z_2(1 + g_{1n}^0(z_2)t)|^2} dH_{2n}(t) \right]^{1/2} \\ &= \left[ \frac{\Im(z_1 g_{1n}^0(z_1)) \Im g_{2n}^0(z_1) - \nu \int \frac{t}{|z_1(1 + g_{1n}^0(z_1)t)|^2} dH_{2n}(t)}{\Im g_{2n}^0(\bar{z}_1) \Im(z_1 g_{1n}^0(\bar{z}_1))} \right]^{1/2} \\ &\quad \times \left[ \frac{\Im(z_2 g_{1n}^0(z_2)) \Im g_{2n}^0(z_2) - \nu \int \frac{t}{|z_2(1 + g_{1n}^0(z_2)t)|^2} dH_{2n}(t)}{\Im g_{2n}^0(\bar{z}_2) \Im(z_2 g_{1n}^0(\bar{z}_2))} \right]^{1/2} < 1. \end{aligned}$$

Using (3.26), (3.14) can be rewritten as for large n

$$\frac{1}{N z_1 z_2} \frac{z_1 g_{2n}^0(z_1) - z_2 g_{2n}^0(z_2)}{g_{1n}^0(z_1) - g_{1n}^0(z_2)} \sum_{k=1}^n r_{nk}(z_1, z_2) \left( 1 - \frac{k-1}{n} f_n(z_1, z_2) \right)^{-1} + o_p(1).$$

Applying Lemma B.4 and (3.24), we have

$$\begin{aligned} &\frac{1}{N} \sum_{k=1}^n r_{nk}(z_1, z_2) \left( 1 - \frac{k-1}{n} f_n(z_1, z_2) \right)^{-1} \\ &= (1 - f_n(z_1, z_2))^{-1} \frac{1}{N} \sum_{k=1}^n r_{nk}(z_1, z_2) - \frac{1}{N} \sum_{k=1}^n \sum_{j=1}^k r_{nj}(z_1, z_2) \\ &\quad \times \left[ \frac{1}{1 - n^{-1} k f_n(z_1, z_2)} - \frac{1}{1 - n^{-1} (k-1) f_n(z_1, z_2)} \right] \\ &= (1 - f_n(z_1, z_2))^{-1} \frac{z_1 g_{1n}^0(z_1) - z_2 g_{1n}^0(z_2)}{g_{2n}^0(z_1) - g_{2n}^0(z_2)} - f_n(z_1, z_2) \frac{1}{N} \sum_{k=1}^n \frac{\sum_{j=1}^k r_{nj}(z_1, z_2)}{k} \\ &\quad \times \frac{n^{-1} k}{(1 - n^{-1} k f_n(z_1, z_2))(1 - n^{-1} (k-1) f_n(z_1, z_2))}. \end{aligned}$$

We next develop the above limit by Abel's lemma. To this end, consider the following two cases, for any  $\varepsilon \in (0, 1/100)$  and large n.

Case 1 : When  $k \leq n^{1-\varepsilon}$ , one gets

$$\begin{aligned} & \frac{1}{N} \sum_{k \leq n^{1-\varepsilon}} \left| \left[ \frac{\sum_{j=1}^k r_{nj}(z_1, z_2)}{k} - c_n^{-1} \frac{z_1 g_{1n}^0(z_1) - z_2 g_{1n}^0(z_2)}{g_{2n}^0(z_1) - g_{2n}^0(z_2)} \right] \right. \\ & \quad \left. \times \frac{n^{-1} k f_n(z_1, z_2)}{(1 - n^{-1} k f_n(z_1, z_2))(1 - n^{-1}(k-1)f_n(z_1, z_2))} \right| \leq CN^{-\varepsilon} = o(1). \end{aligned}$$

Case 2 : When  $k > n^{1-\varepsilon}$ , one gets by (3.24)

$$\begin{aligned} & \frac{1}{N} \sum_{k \geq n^{1-\varepsilon}} \left| \left[ \frac{\sum_{j=1}^k r_{nj}(z_1, z_2)}{k} - c_n^{-1} \frac{z_1 g_{1n}^0(z_1) - z_2 g_{1n}^0(z_2)}{g_{2n}^0(z_1) - g_{2n}^0(z_2)} \right] \right. \\ & \quad \left. \times \frac{n^{-1} k f_n(z_1, z_2)}{(1 - n^{-1} k f_n(z_1, z_2))(1 - n^{-1}(k-1)f_n(z_1, z_2))} \right| \\ & \leq \frac{C}{N} \sum_{k \geq n^{1-\varepsilon}} \left| \frac{\sum_{j=1}^k r_{nj}(z_1, z_2)}{k} - c_n^{-1} \frac{z_1 g_{1n}^0(z_1) - z_2 g_{1n}^0(z_2)}{g_{2n}^0(z_1) - g_{2n}^0(z_2)} \right| = o(1). \end{aligned}$$

Hence, (3.14) can be transformed into

$$\begin{aligned} & f_n(z_1, z_2) (1 - f_n(z_1, z_2))^{-1} - f_n^2(z_1, z_2) \\ & \times \frac{1}{n} \sum_{k=1}^n \frac{n^{-1} k}{(1 - n^{-1} k f_n(z_1, z_2))(1 - n^{-1}(k-1)f_n(z_1, z_2))} + o_p(1). \end{aligned}$$

Thus,

$$(3.14) \xrightarrow{i.p.} \frac{f(z_1, z_2)}{1 - f(z_1, z_2)} - f^2(z_1, z_2) \int_0^1 \frac{t}{(1 - t f(z_1, z_2))^2} dt = \int_0^{f(z_1, z_2)} \frac{1}{1 - z} dz.$$

We conclude that

$$\Phi(z_1, z_2) \xrightarrow{i.p.} \frac{\partial^2}{\partial z_2 \partial z_1} \int_0^{f(z_1, z_2)} \frac{1}{1 - z} dz.$$

**3.2. Tightness of  $M_{n1}(z)$ .** This section is to prove tightness of the sequence of random functions  $\widehat{M}_{n1}(z)$  for  $z \in C$  defined in (2.8). Similar to Section 3 of Bai and Silverstein (2004) (see [1]), it suffices to show that

$$\sup_{n; z_1, z_2 \in C_n} \frac{\mathbb{E} |M_{n1}(z_1) - M_{n1}(z_2)|^2}{|z_1 - z_2|^2}$$

is finite.

We claim that the moments of  $\|\mathbf{D}^{-1}(z)\|$ ,  $\|\mathbf{D}_j^{-1}(z)\|$ , and  $\|\mathbf{D}_{jk}^{-1}(z)\|$  are bounded in  $n$  and  $z \in C_n$ . Without loss of generality, we only give the proof for  $\mathbb{E}\|\mathbf{D}_1^{-1}(z)\|^p$  and the others are similar. In fact, it is obvious for  $z = u + iv \in C_u$ . For  $z \in C_l$  or  $z \in C_r$ , using (3.1) and (3.2), we have for any positive  $p$  and suitably large  $l$

$$\mathbb{E}\|\mathbf{D}_1^{-1}(z)\|^p = \mathbb{E}\|\mathbf{D}_1^{-1}(z)\|^p I(\eta_l \leq \lambda^{\mathbf{B}_1(z)} \leq \eta_r)$$

$$\begin{aligned}
& + \mathbb{E} \|\mathbf{D}_1^{-1}(z)\|^p I(\lambda_{\min}^{\mathbf{B}_1(z)} < \eta_l \text{ or } \lambda_{\max}^{\mathbf{B}_1(z)} > \eta_r) \\
& \leq \max\left\{\frac{1}{|x_r - \eta_r|^p}, \frac{1}{|\eta_l - x_l|^p}\right\} + v^{-p} \mathbb{P}(\lambda_{\min}^{\mathbf{B}_1(z)} < \eta_l \text{ or } \lambda_{\max}^{\mathbf{B}_1(z)} > \eta_r) \\
& \leq C_1 + C_2 n^p \varepsilon_n^{-p} n^{-l} \leq C_p.
\end{aligned}$$

Write

$$m_n(z_1) - m_n(z_2) = \frac{1}{N} \text{tr}(\mathbf{D}^{-1}(z_1) - \mathbf{D}^{-1}(z_2)) = \frac{1}{N} (z_1 - z_2) \text{tr} \mathbf{D}^{-1}(z_1) \mathbf{D}^{-1}(z_2).$$

We then have

$$\begin{aligned}
\frac{M_n(z_1) - M_n(z_2)}{z_1 - z_2} &= \sum_{j=1}^N (\mathbf{E}_j - \mathbf{E}_{j-1}) \text{tr} \mathbf{D}^{-1}(z_1) \mathbf{D}^{-1}(z_2) \\
&= \sum_{j=1}^N (\mathbf{E}_j - \mathbf{E}_{j-1}) \text{tr} (\mathbf{D}^{-1}(z_1) \mathbf{D}^{-1}(z_2) - \mathbf{D}_j^{-1}(z_1) \mathbf{D}_j^{-1}(z_2)) \\
&= \sum_{j=1}^N (\mathbf{E}_j - \mathbf{E}_{j-1}) \text{tr} (\mathbf{D}^{-1}(z_1) - \mathbf{D}_j^{-1}(z_1)) (\mathbf{D}^{-1}(z_2) - \mathbf{D}_j^{-1}(z_2)) \\
&\quad + \sum_{j=1}^N (\mathbf{E}_j - \mathbf{E}_{j-1}) \text{tr} (\mathbf{D}^{-1}(z_1) - \mathbf{D}_j^{-1}(z_1)) \mathbf{D}_j^{-1}(z_2) \\
&\quad + \sum_{j=1}^N (\mathbf{E}_j - \mathbf{E}_{j-1}) \text{tr} \mathbf{D}_j^{-1}(z_1) (\mathbf{D}^{-1}(z_2) - \mathbf{D}_j^{-1}(z_2)) \\
&= \frac{1}{N^2} \sum_{j=1}^N s_j^2 (\mathbf{E}_j - \mathbf{E}_{j-1}) \beta_j(z_1) \beta_j(z_2) (\mathbf{y}_j' \mathbf{D}_j^{-1}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{y}_j)^2 \\
&\quad - \frac{1}{N} \sum_{j=1}^N s_j (\mathbf{E}_j - \mathbf{E}_{j-1}) \beta_j(z_1) \mathbf{y}_j' \mathbf{D}_j^{-2}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{y}_j \\
&\quad - \frac{1}{N} \sum_{j=1}^N s_j (\mathbf{E}_j - \mathbf{E}_{j-1}) \beta_j(z_2) \mathbf{y}_j' \mathbf{D}_j^{-1}(z_1) \mathbf{D}_j^{-2}(z_2) \mathbf{y}_j \\
&\triangleq \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3.
\end{aligned}$$

Thus, it suffices to show that  $\mathbb{E} |\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3|^2$  is bounded. Denote  $\rho_j(z) = \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{y}_j - \text{Etr}(\mathbf{T}_{2n} \mathbf{D}_j^{-1}(z))$ . Note that

$$(3.27) \quad \beta_j(z) = b_j(z) - \frac{1}{N} s_j \beta_j(z) b_j(z) \rho_j(z)$$

$$(3.28) \quad = b_j(z) - \frac{1}{N} s_j b_j^2(z) \rho_j(z) + \frac{1}{N^2} s_j^2 \beta_j(z) b_j^2(z) \rho_j^2(z).$$

Applying (3.27), Lemma B.1, and Lemma A.3, we deduce for all large  $n$

$$\begin{aligned} |b_j(z)| &\leq |\mathbb{E}\beta_j(z)| + \frac{1}{N} |s_j b_j(z) \mathbb{E}(\beta_j(z) \rho_j(z))| \\ &\leq C_1 + C_2 |b_j(z)| N^{-1/2} \leq \frac{C_1}{1 - C_2 N^{-1/2}}. \end{aligned}$$

Hence  $|b_j(z)|$  is bounded for all  $n$ . Using (3.27), write

$$\begin{aligned} \mathcal{P}_1 &= \frac{1}{N^2} \sum_{j=1}^N s_j^2 b_j(z_1) b_j(z_2) (\mathbb{E}_j - \mathbb{E}_{j-1}) (\mathbf{y}_j' \mathbf{D}_j^{-1}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{y}_j)^2 \\ &\quad - \frac{1}{N^3} \sum_{j=1}^N s_j^3 b_j(z_1) b_j(z_2) (\mathbb{E}_j - \mathbb{E}_{j-1}) \beta_j(z_2) \rho_j(z_2) (\mathbf{y}_j' \mathbf{D}_j^{-1}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{y}_j)^2 \\ &\quad - \frac{1}{N^3} \sum_{j=1}^N s_j^3 b_j(z_1) (\mathbb{E}_j - \mathbb{E}_{j-1}) \beta_j(z_1) \beta_j(z_2) \rho_j(z_1) (\mathbf{y}_j' \mathbf{D}_j^{-1}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{y}_j)^2 \\ &\triangleq \mathcal{P}_{11} + \mathcal{P}_{12} + \mathcal{P}_{13}. \end{aligned}$$

By Lemma B.1, we deduce that

$$\begin{aligned} \mathbb{E}|\mathcal{P}_{11}|^2 &= \frac{1}{N^4} \mathbb{E} \left| \sum_{j=1}^N s_j^2 b_j(z_1) b_j(z_2) (\mathbb{E}_j - \mathbb{E}_{j-1}) \left[ (\mathbf{y}_j' \mathbf{D}_j^{-1}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{y}_j)^2 \right. \right. \\ &\quad \left. \left. - (\text{tr} \mathbf{D}_j^{-1}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{T}_{2n})^2 \right] \right|^2 \\ &\leq \frac{C}{N^4} \sum_{j=1}^N \mathbb{E} |\mathbf{y}_j' \mathbf{D}_j^{-1}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{y}_j - \text{tr} \mathbf{D}_j^{-1}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{T}_{2n}|^4 \\ &\quad + \frac{C}{N^2} \sum_{j=1}^N \mathbb{E} |\mathbf{y}_j' \mathbf{D}_j^{-1}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{y}_j - \text{tr} \mathbf{D}_j^{-1}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{T}_{2n}|^2 \\ &\leq \frac{C}{N} + C \leq C. \end{aligned}$$

Using Lemma A.3 and Lemma B.1, one finds

$$\begin{aligned} \mathbb{E}|\mathcal{P}_{12}|^2 &= \frac{1}{N^6} \mathbb{E} \left| \sum_{j=1}^N s_j^3 b_j(z_1) b_j(z_2) (\mathbb{E}_j - \mathbb{E}_{j-1}) \beta_j(z_2) \rho_j(z_2) (\mathbf{y}_j' \mathbf{D}_j^{-1}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{y}_j)^2 \right|^2 \\ &\leq \frac{C}{N^6} \sum_{j=1}^N \mathbb{E} \left| \beta_j(z_2) \rho_j(z_2) (\mathbf{y}_j' \mathbf{D}_j^{-1}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{y}_j - \text{tr} \mathbf{D}_j^{-1}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{T}_{2n})^2 \right|^2 \end{aligned}$$



$$\begin{aligned}
& + \frac{C}{N^2} \sum_{j=1}^N \mathbb{E} |\beta_j(z_2) \rho_j(z_2)|^2 \\
& \leq \frac{C}{N^6} \sum_{j=1}^N \mathbb{E}^{1/2} \left| \rho_j(z_2) (\mathbf{y}_j' \mathbf{D}_j^{-1}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{y}_j - \text{tr} \mathbf{D}_j^{-1}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{T}_{2n}) \right|^2 \Big|^4 \\
& \quad \times \mathbb{E}^{1/2} |\beta_j(z_2)|^4 + \frac{C}{N^2} \sum_{j=1}^N \mathbb{E}^{1/2} |\beta_j(z_2)|^4 \mathbb{E}^{1/2} |\rho_j(z_2)|^4 \\
& \leq \frac{C}{N^2} + C \leq C.
\end{aligned}$$

By the same argument, we get  $\mathbb{E} |\mathcal{P}_{13}|^2 \leq C$ . Hence, we obtain

$$\mathbb{E} |\mathcal{P}_1|^2 \leq C.$$

For  $\mathcal{P}_2$  and  $\mathcal{P}_3$ , we only need to analyze one of them due to their similarity. From (3.27), it is obvious that

$$\begin{aligned}
\mathcal{P}_2 &= -\frac{1}{N} \sum_{j=1}^N s_j b_j(z_1) (\mathbf{E}_j - \mathbf{E}_{j-1}) \mathbf{y}_j' \mathbf{D}_j^{-2}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{y}_j \\
& \quad + \frac{1}{N^2} \sum_{j=1}^N s_j^2 b_j(z_1) (\mathbf{E}_j - \mathbf{E}_{j-1}) \beta_j(z_1) \rho_j(z_1) \mathbf{y}_j' \mathbf{D}_j^{-2}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{y}_j.
\end{aligned}$$

This yields that

$$\begin{aligned}
\mathbb{E} |\mathcal{P}_2|^2 &= \frac{1}{N^2} \mathbb{E} \left| \sum_{j=1}^N s_j b_j(z_1) (\mathbf{E}_j - \mathbf{E}_{j-1}) \mathbf{y}_j' \mathbf{D}_j^{-2}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{y}_j \right|^2 \\
& \quad + \frac{1}{N^4} \mathbb{E} \left| \sum_{j=1}^N s_j^2 b_j(z_1) (\mathbf{E}_j - \mathbf{E}_{j-1}) \beta_j(z_1) \rho_j(z_1) \mathbf{y}_j' \mathbf{D}_j^{-2}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{y}_j \right|^2 \\
& \leq \frac{C}{N^2} \sum_{j=1}^N \mathbb{E} |\mathbf{y}_j' \mathbf{D}_j^{-2}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{y}_j - \text{tr} \mathbf{D}_j^{-2}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{T}_{2n}|^2 \\
& \quad + \frac{C}{N^4} \sum_{j=1}^N \mathbb{E} |\beta_j(z_1) \rho_j(z_1) \mathbf{y}_j' \mathbf{D}_j^{-2}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{y}_j|^2 \\
& \leq C + \frac{C}{N^4} \sum_{j=1}^N \mathbb{E} \left| \beta_j(z_1) \rho_j(z_1) (\mathbf{y}_j' \mathbf{D}_j^{-2}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{y}_j - \text{tr} \mathbf{D}_j^{-2}(z_1) \mathbf{D}_j^{-1}(z_2) \mathbf{T}_{2n}) \right|^2 \\
& \quad + \frac{C}{N^2} \sum_{j=1}^N \mathbb{E} |\beta_j(z_1) \rho_j(z_1)|^2 \leq C
\end{aligned}$$

where the first inequality is from Lemma B.1 and the last inequality is from Lemma A.3. Therefore, we conclude that

$$\sup_{n; z_1, z_2 \in C_n} \frac{\mathbb{E} |M_{n1}(z_1) - M_{n1}(z_2)|^2}{|z_1 - z_2|^2} \leq \sup_{n; z_1, z_2 \in C_n} \mathbb{E} |\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3|^2 \leq C.$$

This implies that  $\widehat{M}_{n1}(z)$  is tight.

**3.3. Convergence of  $M_{n2}(z)$ .** Let  $\mathbf{W}(z) = \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) \mathbf{T}_{2n} - z \mathbf{I}$ . Our first aim is to prove that  $\|\mathbf{W}^{-1}(z)\|$  is uniformly bounded on  $C_n$ . Indeed we have for any positive number  $t \geq 0$

$$\begin{aligned} \Im\left(\frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) t - z\right) &= \frac{1}{N} \sum_{j=1}^n \frac{s_j^2 t}{|1 + s_j \mathbb{E} g_{2n}(z)|^2} \mathbb{E} \Im g_{2n}(\bar{z}) - v \\ &= -v \left(1 + \frac{1}{N^2} \sum_{j \neq k} \mathbb{E} \frac{s_j^2 t}{|1 + s_j \mathbb{E} g_{2n}(z)|^2} \text{Etr}(\mathbf{D}^{-1}(z) \mathbf{D}^{-1}(\bar{z}) \mathbf{T}_{2n})\right) \leq -v \end{aligned}$$

which yields  $\|\mathbf{W}^{-1}(z)\|$  is bounded by  $v_0^{-1}$  on  $C_u$ . Since  $\Im(z g_1(z)) > 0$ , there exists a positive constant  $\delta_1$  such that for any  $t$  in the support of  $H_2$

$$\inf_{z \in C_l \cup C_r} |z g_1(z) t + z| \geq \delta_1.$$

Moreover, since  $z g_1(z)$  is continuous on  $C_l \cup C_r$ , there exists  $C_0 > 0$  such that

$$\sup_{z \in C_l \cup C_r} |z g_1(z)| < C_0.$$

Additionally, using  $H_{2n} \xrightarrow{d} H_2$ , for all large  $n$  and any  $t$  in the support of  $H_2$ , there exists an eigenvalue  $\lambda^{\mathbf{T}_{2n}}$  of  $\mathbf{T}_{2n}$  such that

$$|\lambda^{\mathbf{T}_{2n}} - t| \leq \frac{\delta_1}{4C_0}.$$

Assume for the moment that

$$(3.29) \quad \sup_{z \in C_l \cup C_r} \left| \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) + z g_1(z) \right| < \frac{\delta_1}{4\tau}.$$

It follows that

$$\begin{aligned} \inf_{z \in C_l \cup C_r} \left| \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) \lambda^{\mathbf{T}_{2n}} - z \right| &\geq \inf_{z \in C_l \cup C_r} |z g_1(z) t + z| \\ &- \sup_{z \in C_l \cup C_r} |z g_1(z)| |\lambda^{\mathbf{T}_{2n}} - t| - \sup_{z \in C_l \cup C_r} |\lambda^{\mathbf{T}_{2n}}| \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) + z g_1(z) \geq \delta/2. \end{aligned}$$

We conclude that

$$(3.30) \quad \sup_{n, C_n} \|\mathbf{W}^{-1}(z)\| < \infty.$$

We are now in position to prove (3.29), i.e.,

$$\sup_{z \in C_l \cup C_r} \left| \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) + z g_1(z) \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By (3.1), (3.2) and (3.3), we find

$$\begin{aligned} \sup_{z \in C_l \cup C_r} |\mathbb{E} g_{2n}(z)| &= \sup_{z \in C_l \cup C_r} \left| \frac{1}{N} \mathbb{E} \left[ \text{tr}(\mathbf{D}^{-1} \mathbf{T}_{2n}) I(\eta_l \leq \lambda^{\mathbf{B}_n} \leq \eta_r) \right] \right| \\ &+ \sup_{z \in C_l \cup C_r} \left| \frac{1}{N} \mathbb{E} \left[ \text{tr}(\mathbf{D}^{-1} \mathbf{T}_{2n}) I(\lambda_{\min}^{\mathbf{B}_n} < \eta_l \text{ or } \lambda_{\max}^{\mathbf{B}_n} > \eta_r) \right] \right| \\ &\leq \tau \max \left\{ \frac{1}{x_l - \eta_l}, \frac{1}{\eta_r - x_r} \right\} + \tau n^{1+\alpha} \mathbb{P}(\lambda_{\min}^{\mathbf{B}_n} < \eta_l \text{ or } \lambda_{\max}^{\mathbf{B}_n} > \eta_r) \\ &< 1/2\tau + 1/4\tau = 3/4\tau \end{aligned}$$

which implies that  $\psi_j(z)$  is bounded on  $C_l \cup C_r$ . For  $z \in C_l \cup C_r$ , rewrite

$$\begin{aligned} \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) + z g_1(z) &= (c_n - c) \int \frac{x}{1 + x \mathbb{E} g_{2n}(z)} dH_{1n}(x) \\ &- c(\mathbb{E} g_{2n}(z) - g_2(z)) \int \frac{x^2}{(1 + x \mathbb{E} g_{2n}(z))(1 + x g_2(z))} dH_{1n}(x) \\ &+ c \int \frac{x}{1 + x g_2(z)} d(H_{1n}(x) - dH_1(x)). \end{aligned}$$

Using Lemma A.2, one gets

$$(3.31) \quad \sup_{z \in C_l \cup C_r} |\mathbb{E} g_{2n}(z) - g_2(z)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since the functions  $x/(1+xg_2(z))$  in  $x \in \{s_1, \dots, s_n\}$  form a bounded, equicontinuous family  $z$  ranges in  $C_l \cup C_r$ , by Problem 8, page 17 in [4]] and the fact that  $H_{1n} \rightarrow H_1$  we find that

$$(3.32) \quad \sup_{z \in C_l \cup C_r} \left| \int \frac{x}{1 + x g_2(z)} d(H_{1n}(x) - dH_1(x)) \right| \rightarrow 0$$

Combining (3.31) with (3.32) we obtain (3.29).

Write  $\mathbf{D}(z) - \mathbf{W}(z) = \frac{1}{N} \sum_{j=1}^n s_j \mathbf{y}_j \mathbf{y}_j' - \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) \mathbf{T}_{2n}$ . Taking inverses and then expected value, we have

$$\begin{aligned} (3.33) \quad &\mathbf{W}^{-1}(z) - \mathbf{E} \mathbf{D}^{-1}(z) \\ &= \mathbf{W}^{-1}(z) \mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^n s_j \mathbf{y}_j \mathbf{y}_j' \mathbf{D}^{-1}(z) - \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \right] \\ &= \mathbf{W}^{-1}(z) \mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^n s_j \beta_j(z) \mathbf{y}_j \mathbf{y}_j' \mathbf{D}_j^{-1}(z) - \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \right]. \end{aligned}$$

Taking the trace on both sides and dividing by  $-1$ , one obtains

(3.34)

$$\begin{aligned}
d_{n1}(z) &= -\frac{1}{N} \sum_{j=1}^n s_j E\beta_j(z) \left( \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{W}^{-1}(z) \mathbf{y}_j - \text{Etr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \right) \right) \\
&\quad - \frac{1}{N} \sum_{j=1}^n s_j E\beta_j(z) \left( \text{Etr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \right) - \text{Etr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \right) \right) \\
&\quad - \frac{1}{N} \sum_{j=1}^n s_j E \left( \beta_j(z) - \psi_j(z) \right) E \left( \text{tr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \right) \right) \\
&\triangleq \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3,
\end{aligned}$$

where  $d_{n1}(z) = N \left[ E m_n(z) - \int \frac{1}{\frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) x - z} dH_{2n}(x) \right]$ . From (3.28),  $\mathcal{J}_1$  can be represented as

$$\begin{aligned}
\mathcal{J}_1 &= \frac{1}{N^2} \sum_{j=1}^n s_j^2 b_j^2(z) E \rho_j(z) \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{W}^{-1}(z) \mathbf{y}_j \\
&\quad - \frac{1}{N^3} \sum_{j=1}^n s_j^3 b_j^2 E \beta_j(z) \rho_j^2(z) \left( \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{W}^{-1}(z) \mathbf{y}_j - \text{tr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \right) \right) \\
&\quad - \frac{1}{N^3} \sum_{j=1}^n s_j^3 b_j^2 E \beta_j(z) \rho_j^2(z) \left( \text{tr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \right) - \text{Etr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \right) \right) \\
&\triangleq \mathcal{J}_{11} + \mathcal{J}_{12} + \mathcal{J}_{13}.
\end{aligned}$$

Applying Lemma B.1, Lemma A.3, and (3.30), we get

$$\begin{aligned}
|\mathcal{J}_{12}| &\leq \frac{C}{N^3} \sum_{j=1}^n \left( E |\beta_j(z)|^4 \right)^{1/4} \left( E |\rho_j(z)|^8 \right)^{1/4} \\
&\quad \times \left( E \left| \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{W}^{-1}(z) \mathbf{y}_j - \text{tr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \right) \right|^2 \right)^{1/2} \\
&\leq \frac{C}{\sqrt{N}} \rightarrow 0
\end{aligned}$$

and

$$\begin{aligned}
|\mathcal{J}_{13}| &\leq \frac{C}{N^3} \sum_{j=1}^n \left( E |\beta_j(z)|^4 \right)^{1/4} \left( E |\rho_j(z)|^8 \right)^{1/4} \\
&\quad \times \left( E \left| \text{tr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \right) - \text{Etr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \right) \right|^2 \right)^{1/2} \\
&\leq \frac{C}{N} \rightarrow 0
\end{aligned}$$

where the last inequality is obtained from Lemma B.5. Moreover, we have

$$\begin{aligned}\mathcal{J}_{11} &= \frac{1}{N^2} \sum_{j=1}^n s_j^2 b_j^2(z) \mathbb{E} \varepsilon_j(z) \left( \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{W}^{-1}(z) \mathbf{y}_j - \text{tr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \right) \right) \\ &\quad + \frac{1}{N^2} \sum_{j=1}^n s_j^2 b_j^2(z) \text{Cov} \left( \text{tr} \left( \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \right), \text{tr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \right) \right) \\ &\triangleq \mathcal{J}_{111} + \mathcal{J}_{112}.\end{aligned}$$

Using Lemma B.5 and the Cauchy-Swcharz inequality, one finds  $|\mathcal{J}_{112}| \leq CN^{-1}$ . This yields

(3.35)

$$\mathcal{J}_1 = \frac{1}{N^2} \sum_{j=1}^n s_j^2 b_j^2(z) \mathbb{E} \varepsilon_j(z) \left( \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{W}^{-1}(z) \mathbf{y}_j - \text{tr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \right) \right) + o(1).$$

Note that (3.27) and

$$\begin{aligned}&\text{Etr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \right) - \text{Etr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \right) \\ &= \frac{1}{N} \sum_{j=1}^n s_j \mathbb{E} \beta_j(z) \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{y}_j.\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{J}_2 &= -\frac{1}{N^2} \sum_{j=1}^n s_j^2 \mathbb{E} \beta_j(z) \mathbb{E} \beta_j(z) \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{y}_j \\ &= -\frac{1}{N^2} \sum_{j=1}^n s_j^2 b_j^2(z) \mathbb{E} \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{y}_j \\ &\quad + \frac{1}{N^3} \sum_{j=1}^n s_j^3 b_j^2(z) \mathbb{E} \beta_j(z) \rho_j(z) \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{y}_j \\ &\quad + \frac{1}{N^3} \sum_{j=1}^n s_j^3 b_j(z) \mathbb{E} \beta_j(z) \rho_j(z) \mathbb{E} \beta_j(z) \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{y}_j \\ &\triangleq \mathcal{J}_{21} + \mathcal{J}_{22} + \mathcal{J}_{23}.\end{aligned}$$

From Lemma B.1 and (3.30), we see  $|\mathcal{J}_{22} + \mathcal{J}_{23}| \leq \frac{C}{\sqrt{N}}$ . Hence,

$$(3.36) \quad \mathcal{J}_2 = -\frac{1}{N^2} \sum_{j=1}^n s_j^2 b_j^2(z) \text{Etr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \right) + o(1).$$

Note that from (3.28)

$$\mathbb{E} \left( \beta_j(z) - b_j(z) \right) = \frac{1}{N^2} s_j^2 b_j^3(z) \mathbb{E} \rho_j^2(z) - \frac{1}{N^3} s_j^3 b_j^3(z) \mathbb{E} \beta_j(z) \rho_j^3(z)$$

$$\begin{aligned}
&= \frac{1}{N^2} s_j^2 b_j^3(z) E \varepsilon_j^2(z) - \frac{1}{N^3} s_j^3 b_j^3(z) E \beta_j(z) \rho_j^3(z) \\
&\quad + \frac{1}{N^2} s_j^2 b_j^3(z) E \left( \text{tr} \left( \mathbf{D}_j^{-1} \mathbf{T}_{2n} \right) - E \text{tr} \left( \mathbf{D}_j^{-1} \mathbf{T}_{2n} \right) \right)^2 \\
&\triangleq \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3.
\end{aligned}$$

By Lemma B.1, we obtain

$$|\mathcal{H}_2| \leq \frac{C}{N^3} E^{1/2} |\beta_j(z)|^2 E^{1/2} |\rho_j(z)|^6 \leq C N^{-3/2} = o(N^{-1}).$$

Using Lemma B.5, we have

$$|\mathcal{H}_3| \leq C N^{-2} = o(N^{-1}).$$

These imply that

$$E \left( \beta_j(z) - b_j(z) \right) = \frac{1}{N^2} s_j^2 b_j^3(z) E \varepsilon_j^2(z) + o(N^{-1}).$$

Moreover,

$$\begin{aligned}
b_j(z) - \psi_j(z) &= -\frac{1}{N} s_j b_j(z) \psi_j(z) E \left( \text{tr} \left( \mathbf{D}_j^{-1} \mathbf{T}_{2n} \right) - \text{tr} \left( \mathbf{D}^{-1} \mathbf{T}_{2n} \right) \right) \\
&= -\frac{1}{N^2} s_j^2 b_j(z) \psi_j(z) E \beta_j(z) \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1} \mathbf{y}_j \\
&= -\frac{1}{N^2} s_j^2 b_j^2(z) \psi_j(z) E \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1} \mathbf{y}_j \\
&\quad + \frac{1}{N^3} s_j^3 b_j^2(z) \psi_j(z) E \beta_j(z) \rho_j(z) \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1} \mathbf{y}_j.
\end{aligned}$$

From Lemma A.3 and Lemma B.1, we have

$$\begin{aligned}
&\left| \frac{1}{N^3} s_j^3 b_j^2(z) \psi_j(z) E \beta_j(z) \rho_j(z) \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1} \mathbf{y}_j \right| \\
&\leq \frac{1}{N^3} s_j^3 E^{1/2} |\beta_j(z)|^2 E^{1/2} |\rho_j(z) \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1} \mathbf{y}_j|^2 \leq C N^{-3/2}.
\end{aligned}$$

This yields that

$$(3.37) \quad b_j(z) - \psi_j(z) = -\frac{1}{N^2} s_j^2 b_j^2(z) \psi_j(z) E \text{tr} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1} \mathbf{T}_{2n} + o(N^{-1}).$$

Hence,

$$\begin{aligned}
(3.38) \quad E \left( \beta_j(z) - \psi_j(z) \right) &= -\frac{1}{N^2} s_j^2 b_j^2(z) \psi_j(z) E \text{tr} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1} \mathbf{T}_{2n} \\
&\quad + \frac{1}{N^2} s_j^2 b_j^3(z) E \varepsilon_j^2(z) + o(N^{-1}).
\end{aligned}$$

Thus, we get

(3.39)

$$\begin{aligned}\mathcal{J}_3 = & -\frac{1}{N^3} \sum_{j=1}^n s_j^3 b_j^3(z) E \varepsilon_j^2(z) E \left( \text{tr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \right) \right) \\ & + \frac{1}{N^3} s_j^3 b_j^2(z) \psi_j(z) E \text{tr} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} E \left( \text{tr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \right) \right) + o(1).\end{aligned}$$

From (3.34), (3.35), (3.36) and (3.39), we conclude that

$$\begin{aligned}d_{n1}(z) = & \frac{1}{N^2} \sum_{j=1}^n s_j^2 b_j^2(z) E \varepsilon_j(z) \left( \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{W}^{-1}(z) \mathbf{y}_j - \text{tr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \right) \right) \\ & - \frac{1}{N^2} \sum_{j=1}^n s_j^2 b_j^2(z) E \text{tr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \right) \\ & - \frac{1}{N^3} \sum_{j=1}^n s_j^3 \psi_j b_j^2(z) E \varepsilon_j^2(z) E \left( \text{tr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \right) \right) \\ & + \frac{1}{N^3} s_j^3 b_j^2(z) \psi_j(z) E \text{tr} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} E \left( \text{tr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \right) \right) + o(1).\end{aligned}$$

It is evident from (3.37) that

$$(3.40) \quad |b_j(z) - \psi_j(z)| \leq \frac{C}{N} E \left\| \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \right\| + o(N^{-1}) \leq \frac{C}{N}.$$

Then,

$$\begin{aligned}d_{n1}(z) = & \frac{1}{N^2} \sum_{j=1}^n s_j^2 \psi_j^2(z) E \varepsilon_j(z) \left( \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{W}^{-1}(z) \mathbf{y}_j - \text{tr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \right) \right) \\ & - \frac{1}{N^2} \sum_{j=1}^n s_j^2 \psi_j^2(z) E \text{tr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \right) \\ & - \frac{1}{N^3} \sum_{j=1}^n s_j^3 \psi_j^3(z) E \varepsilon_j^2(z) E \left( \text{tr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \right) \right) \\ & + \frac{1}{N^3} s_j^3 \psi_j^3(z) E \text{tr} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} E \left( \text{tr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \right) \right) + o(1).\end{aligned}$$

Considering the moments of the Gaussian variables, we have

$$\begin{aligned}d_{n1}(z) = & \frac{1}{N^2} \sum_{j=1}^n s_j^2 \psi_j^2(z) E \text{tr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \right) \\ & - \frac{1}{N^3} \sum_{j=1}^n s_j^3 \psi_j^3(z) E \text{tr} \left( \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \right) E \left( \text{tr} \left( \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \right) \right) + o(1).\end{aligned}$$

Write  $M_{n2}(z)$  as

$$\begin{aligned}
& N \left[ E m_n(z) - m_n^0(z) \right] \\
&= d_{n1}(z) + N \left[ \int \frac{1}{\frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) x - z} dH_{2n}(x) + z^{-1} \int \frac{1}{1 + g_{1n}^0(z) x} dH_{2n}(x) \right] \\
&= d_{n1}(z) - N \left( E g_{2n}(z) - g_{2n}^0(z) \right) \frac{1}{N} \sum_{j=1}^n \frac{s_j^2 \psi_j(z)}{1 + g_{2n}^0(z) s_j} \\
&\quad \times \int \frac{x}{\left( \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) x - z \right) \left( z + z g_{1n}^0(z) x \right)} dH_{2n}(x).
\end{aligned}$$

Below we first find the relation between  $(E m_n(z) - m_n^0(z))$  and  $(E g_{2n}(z) - g_{2n}^0(z))$ . Write  $\mathbf{D}(z) + z \mathbf{I}_N = \frac{1}{N} \sum_{k=1}^n s_k \mathbf{y}_k \mathbf{y}_k'$ . Multiplying by  $\mathbf{D}^{-1}(z)$  on the right-hand side and using the formula (3.7), we obtain

$$\begin{aligned}
\mathbf{I}_N + z \mathbf{D}^{-1}(z) &= \frac{1}{N} \sum_{k=1}^n s_k \mathbf{y}_k \mathbf{y}_k' \left( \mathbf{D}_k^{-1}(z) - \frac{1}{N} s_k \beta_k(z) \mathbf{D}_k^{-1}(z) \mathbf{y}_k \mathbf{y}_k' \mathbf{D}_k^{-1}(z) \right) \\
&= \frac{1}{N} \sum_{k=1}^n s_k \beta_k(z) \mathbf{y}_k \mathbf{y}_k' \mathbf{D}_k^{-1}(z).
\end{aligned}$$

Taking the trace on both side and dividing by  $N$ , one gets

$$1 + z m_n(z) = c_n - c_n n^{-1} \sum_{k=1}^n \beta_k(z).$$

Together with (1.4), we have

$$(3.41) \quad \underline{m}_n(z) = -\frac{1}{zn} \sum_{k=1}^n \beta_k(z).$$

It is obtained from (3.38) and (3.40)

$$E \underline{m}_n(z) + \frac{1}{znN^2} \sum_{j=1}^n s_j^2 \psi_j^3(z) \text{Etr} \left( \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \right) = -\frac{1}{zn} \sum_{j=1}^n \psi_j(z) + o(N^{-1}).$$

Thus

$$\begin{aligned}
E \underline{m}_n(z) - \underline{m}_n^0(z) &= -\frac{1}{znN^2} \sum_{j=1}^n s_j^2 \psi_j^3(z) \text{Etr} \left( \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \right) \\
&\quad + \left( E g_{2n}(z) - g_{2n}^0(z) \right) \frac{1}{zn} \sum_{j=1}^n \frac{s_j \psi_j(z)}{1 + g_{2n}^0(z) s_j} + o(N^{-1}).
\end{aligned}$$



Consequently,

$$\begin{aligned} \mathbb{E}g_{2n}(z) - g_{2n}^0(z) &= \left[ \frac{1}{zn} \sum_{j=1}^n \frac{s_j \psi_j(z)}{1 + g_{2n}^0(z) s_j} \right]^{-1} \times \left[ \mathbb{E} \underline{m}_n(z) - \underline{m}_n^0(z) \right. \\ &\quad \left. + \frac{1}{znN^2} \sum_{j=1}^n s_j^2 \psi_j^3(z) \text{Etr} \left( \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \right) \right] + o(N^{-1}). \end{aligned}$$

Combining the above equalities with  $M_{n2}(z) = N [\mathbb{E}m_n(z) - m_n^0(z)] = n [\mathbb{E} \underline{m}_n(z) - \underline{m}_n^0(z)]$ , we conclude that

$$\begin{aligned} (3.42) \quad N [\mathbb{E}m_n(z) - m_n^0(z)] &= d_{n1}(z) - n [\mathbb{E} \underline{m}_n(z) - \underline{m}_n^0(z)] \left[ \frac{1}{zn} \sum_{j=1}^n \frac{s_j \psi_j(z)}{1 + g_{2n}^0(z) s_j} \right]^{-1} \\ &\quad \times \frac{1}{n} \sum_{j=1}^n \frac{s_j^2 \psi_j(z)}{1 + g_{2n}^0(z) s_j} \int \frac{x}{\left( \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) x - z \right) (z + z g_{1n}^0(z) x)} dH_{2n}(x) \\ &\quad - \frac{1}{znN^2} \sum_{j=1}^n s_j^2 \psi_j^3(z) \text{Etr} \left( \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \right) \left[ \frac{1}{zn} \sum_{j=1}^n \frac{s_j \psi_j(z)}{1 + g_{2n}^0(z) s_j} \right]^{-1} \\ &\quad \times \frac{1}{n} \sum_{j=1}^n \frac{s_j^2 \psi_j(z)}{1 + g_{2n}^0(z) s_j} \int \frac{x}{\left( \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) x - z \right) (z + z g_{1n}^0(z) x)} dH_{2n}(x) \\ (3.43) \quad &= (d_{n1}(z) - d_{n2}(z)) \left\{ 1 + z \left[ \frac{1}{n} \sum_{j=1}^n \frac{s_j \psi_j(z)}{1 + g_{2n}^0(z) s_j} \right]^{-1} \frac{1}{n} \sum_{j=1}^n \frac{s_j^2 \psi_j(z)}{1 + g_{2n}^0(z) s_j} \right. \\ &\quad \left. \times \int \frac{x}{\left( \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) x - z \right) (z + z g_{1n}^0(z) x)} dH_{2n}(x) \right\}^{-1} \end{aligned}$$

where

$$\begin{aligned} d_{n2}(z) &= \frac{1}{N^2} \sum_{j=1}^n s_j^2 \psi_j^3(z) \text{Etr} \left( \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \right) \left[ \frac{1}{n} \sum_{j=1}^n \frac{s_j \psi_j(z)}{1 + g_{2n}^0(z) s_j} \right]^{-1} \\ &\quad \times \frac{1}{n} \sum_{j=1}^n \frac{s_j^2 \psi_j(z)}{1 + g_{2n}^0(z) s_j} \int \frac{x}{\left( \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) x - z \right) (z + z g_{1n}^0(z) x)} dH_{2n}(x). \end{aligned}$$

Write

$$\frac{1}{n} \sum_{j=1}^n \frac{s_j}{(1 + g_2(z) s_j)^2} - \frac{z}{n} \sum_{j=1}^n \frac{s_j^2}{(1 + g_2(z) s_j)^2} \int \frac{x}{(z + z g_1(z) x)^2} dH_2(x)$$

$$\begin{aligned}
&= -\frac{zg_1(z)}{c_n} - \frac{1}{n} \sum_{j=1}^n \frac{s_j^2}{(1 + g_2(z)s_j)^2} \left[ g_2(z) + z \int \frac{x}{(z + zg_1(z)x)^2} dH_2(x) \right] \\
&= -\frac{zg_1(z)}{c_n} \left\{ 1 - \frac{1}{z^2 N} \sum_{j=1}^n \frac{s_j^2}{(1 + g_2(z)s_j)^2} \int \frac{x^2}{(1 + g_1(z)x)^2} dH_2(x) \right\}.
\end{aligned}$$

Note that for all  $z = u + iv \in \mathbb{C}$

$$\begin{aligned}
&\left| \frac{1}{z^2 N} \sum_{j=1}^n \frac{s_j^2}{(1 + g_2(z)s_j)^2} \int \frac{x^2}{(1 + g_1(z)x)^2} dH_2(x) \right| \\
&\leq \frac{1}{N} \sum_{j=1}^n \frac{s_j^2}{|1 + g_2(z)s_j|^2} \int \frac{x^2}{|z + zg_1(z)x|^2} dH_2(x) \\
&= \frac{\Im(zg_1(z))}{\Im g_2(z)} \frac{\Im g_2(z) - v \int \frac{x}{|z + zg_1(z)x|^2} dH_2(x)}{\Im(zg_1(z))} < 1.
\end{aligned}$$

By continuity, we have the denominator of (3.43) is bounded away from zero.

We are now in position to find the limits of  $d_{n1}(z)$  and  $d_{n2}(z)$ . Due to (3.7) and (3.30), we see that

$$\begin{aligned}
&\left| \text{Etr}(\mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z)) - \text{Etr}(\mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z)) \right| \\
&\leq \left| \text{Etr}(\mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z)) - \text{Etr}(\mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z)) \right| \\
&\quad + \left| \text{Etr}(\mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z)) - \text{Etr}(\mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z)) \right| \\
&\leq \frac{\tau}{N} \mathbb{E} |\beta_j(z)| \left| \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{y}_j \right| \\
&\quad + \frac{\tau}{N} \mathbb{E} |\beta_j(z)| \left| \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{y}_j \right| \\
&\leq C
\end{aligned}$$

where the last inequality is from Lemma B.1 and Lemma A.3. By the same argument, it follows that

$$\left| \text{Etr}(\mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z)) - \text{Etr}(\mathbf{T}_{2n} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z)) \right| \leq C.$$

Hence,

$$\begin{aligned}
d_{n1}(z) &= \frac{1}{N^2} \sum_{j=1}^n s_j^2 \psi_j^2(z) \text{Etr}(\mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z)) \\
&\quad - \frac{1}{N^3} \sum_{j=1}^n s_j^3 \psi_j^3(z) \text{Etr}(\mathbf{T}_{2n} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z)) \mathbb{E}(\text{tr}(\mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z))) + o(1).
\end{aligned}$$

Our next goal is to find the limit of  $\text{Etr}(\mathbf{T}_{2n}\mathbf{D}^{-1}(z)\mathbf{T}_{2n}\mathbf{D}^{-1}(z))$ ,  $\text{Etr}(\mathbf{W}^{-1}(z)\mathbf{T}_{2n}\mathbf{D}^{-1}(z))$  and  $\text{Etr}(\mathbf{W}^{-1}(z)\mathbf{T}_{2n}\mathbf{D}^{-1}(z)\mathbf{T}_{2n}\mathbf{D}^{-1}(z))$ . From (3.33), we have

$$\begin{aligned}
 (3.44) \quad \mathbf{W}^{-1}(z) - \mathbf{D}^{-1}(z) &= \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) \mathbf{W}^{-1}(z) (\mathbf{y}_j \mathbf{y}_j' - \mathbf{T}_{2n}) \mathbf{D}_j^{-1}(z) \\
 &\quad + \frac{1}{N} \sum_{j=1}^n s_j (\beta_j(z) - \psi_j(z)) \mathbf{W}^{-1}(z) \mathbf{y}_j \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \\
 &\quad + \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) \mathbf{W}^{-1}(z) \mathbf{T}_{2n} (\mathbf{D}_j^{-1}(z) - \mathbf{D}^{-1}(z)) \\
 &\triangleq \mathbf{G}_1(z) + \mathbf{G}_2(z) + \mathbf{G}_3(z).
 \end{aligned}$$

Let  $\mathbf{M}$  be  $N \times N$  matrix with a nonrandom bound on the spectral norm of  $\mathbf{M}$  for all parameters governing  $\mathbf{M}$  and under all realizations of  $\mathbf{M}$ . Applying Lemma B.1, Lemma B.5, and Lemma A.3, we obtain

$$(3.45) \quad \mathbb{E}|\beta_j(z) - \psi_j(z)|^2 \leq \frac{C}{N^2} \mathbb{E}|\beta_j(z) (\mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{y}_j - \mathbf{E} \mathbf{D}^{-1}(z) \mathbf{T}_{2n})|^2 = O(N^{-1})$$

which implies that

$$\begin{aligned}
 (3.46) \quad \mathbb{E}|\text{tr}(\mathbf{G}_2(z)\mathbf{M})| &\leq \frac{C}{N} \sum_{j=1}^n \mathbb{E}^{1/2} |\beta_j(z) - \psi_j(z)|^2 \\
 &\quad \times \mathbb{E}^{1/2} |\mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{M} \mathbf{W}^{-1}(z) \mathbf{y}_j|^2 = O(N^{1/2}).
 \end{aligned}$$

Form Lemma B.1 and (3.7), we have

$$(3.47) \quad \mathbb{E}|\text{tr}(\mathbf{G}_3(z)\mathbf{M})| \leq \frac{C}{N^2} \sum_{j=1}^n |\mathbb{E} \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{y}_j| \leq C.$$

Furthermore, write

$$\begin{aligned}
 &\text{tr}((\mathbf{G}_1(z)) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \mathbf{M}) \\
 &= \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) \text{tr} \mathbf{W}^{-1}(z) \mathbf{y}_j \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} (\mathbf{D}^{-1}(z) - \mathbf{D}_j^{-1}(z)) \mathbf{M} \\
 &\quad + \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) \text{tr} \mathbf{W}^{-1}(z) (\mathbf{y}_j \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{M} - \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{M}) \\
 &\quad + \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) \text{tr} \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} (\mathbf{D}_j^{-1}(z) - \mathbf{D}^{-1}(z)) \mathbf{M} \\
 &= -\frac{1}{N^2} \sum_{j=1}^n s_j^2 \psi_j(z) \beta_j(z) \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{y}_j \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{M} \mathbf{W}^{-1}(z) \mathbf{y}_j
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) \text{tr} \mathbf{W}^{-1}(z) \left( \mathbf{y}_j \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{M} - \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{M} \right) \\
& + \frac{1}{N^2} \sum_{j=1}^n s_j^2 \psi_j(z) \beta_j(z) \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{M} \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{y}_j \\
& \triangleq p_1(z) + p_2(z) + p_3(z).
\end{aligned}$$

It is obvious that  $\mathbb{E} p_2(z) = 0$ . Using Lemma B.1 and Lemma A.3, we have

$$\mathbb{E} |p_3(z)| \leq C.$$

Together with (3.45) and Lemma B.1, one gets

$$\mathbb{E} p_1(z) = -\frac{1}{N^2} \sum_{j=1}^n s_j^2 \psi_j^2(z) \mathbb{E} \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{y}_j \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{M} \mathbf{W}^{-1}(z) \mathbf{y}_j + o(1).$$

By the proof of Lemma A.2, we obtain

$$|\psi_j(z) - \frac{1}{1 + s_j g_{2n}^0(z)}| = o(1).$$

Let  $q_j = \frac{s_j^2}{(1 + s_j g_{2n}^0(z))^2}$ . Then, combining Lemma B.1, we find

$$\begin{aligned}
\mathbb{E} p_1(z) &= -\frac{1}{N^2} \sum_{j=1}^n q_j \mathbb{E} \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{y}_j \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{M} \mathbf{W}^{-1}(z) \mathbf{y}_j + o(N) \\
&= -\frac{1}{N^2} \sum_{j=1}^n q_j \mathbb{E} \left( \text{tr} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \mathbf{D}_j^{-1}(z) \mathbf{T}_{2n} \right) \left( \text{tr} \mathbf{D}_j^{-1}(z) \mathbf{M} \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \right) + o(N) \\
&= -\frac{1}{N^2} \sum_{j=1}^n q_j \mathbb{E} \left( \text{tr} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \right) \left( \text{tr} \mathbf{D}^{-1}(z) \mathbf{M} \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \right) + o(N) \\
&= -\frac{1}{N^2} \sum_{j=1}^n q_j \left( \mathbb{E} \text{tr} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \right) \left( \mathbb{E} \text{tr} \mathbf{D}^{-1}(z) \mathbf{M} \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \right) + o(N).
\end{aligned}$$

It follows that

$$\begin{aligned}
(3.48) \quad & \mathbb{E} \text{tr}((\mathbf{G}_1(z)) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \mathbf{M}) \\
&= -\frac{1}{N^2} \sum_{j=1}^n q_j \left( \mathbb{E} \text{tr} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \right) \left( \mathbb{E} \text{tr} \mathbf{D}^{-1}(z) \mathbf{M} \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \right) + o(N).
\end{aligned}$$

From (3.44), (3.46)-(3.48), and

$$\mathbb{E} \text{tr}((\mathbf{G}_1(z)) \mathbf{T}_{2n} \mathbf{W}^{-1}(z) \mathbf{T}_{2n}) = 0,$$

one has

$$\begin{aligned}
 \frac{1}{N} \text{Etr} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \mathbf{W}^{-1}(z) \mathbf{T}_{2n} &= \frac{1}{N} \text{tr} \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{W}^{-1}(z) \mathbf{T}_{2n} + o(1) \\
 (3.49) \quad &= \int \frac{t^2}{\left( \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) t - z \right)^2} dH_{2n}(t) + o(1).
 \end{aligned}$$

By the same argument, we get

$$\begin{aligned}
 \frac{1}{N} \text{Etr} \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) &= \frac{1}{N} \text{tr} \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{W}^{-1}(z) + o(1) \\
 &= \int \frac{t^2}{\left( \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) t - z \right)^3} dH_{2n}(t) + o(1).
 \end{aligned}$$

and

$$\begin{aligned}
 (3.50) \quad \frac{1}{N} \text{Etr} \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) &= \frac{1}{N} \text{tr} \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{W}^{-1}(z) + o(1) \\
 &= \int \frac{t}{\left( \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) t - z \right)^2} dH_{2n}(t) + o(1).
 \end{aligned}$$

Together with (3.44), (3.46)-(3.49), we have

$$\begin{aligned}
 \frac{1}{N} \text{Etr} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} &= \frac{1}{N} \text{Etr} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \\
 &\quad \times \left[ 1 + \frac{1}{N^2} \sum_{j=1}^n q_j \left( \text{Etr} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \right) \right] + o(1) \\
 &= \int \frac{t^2}{\left( \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) t - z \right)^2} dH_{2n}(t) \\
 &\quad \times \left[ 1 + \frac{1}{N^2} \sum_{j=1}^n q_j \left( \text{Etr} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \right) \right] + o(1)
 \end{aligned}$$

which yields

$$\begin{aligned}
 (3.51) \quad \frac{1}{N} \text{Etr} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} &= \int \frac{t^2}{\left( \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) t - z \right)^2} dH_{2n}(t) \\
 &\quad \times \left[ 1 - \frac{1}{N} \sum_{j=1}^n q_j \int \frac{t^2}{\left( \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) t - z \right)^2} dH_{2n}(t) \right]^{-1} + o(1).
 \end{aligned}$$

Similarly, we get

(3.52)

$$\begin{aligned}
& \frac{1}{N} \text{Etr}(\mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z)) \\
&= \frac{1}{N} \text{Etr} \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{W}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \left[ 1 + \frac{1}{N^2} \sum_{j=1}^n q_j \left( \text{Etr} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \right) \right] \\
&= \int \frac{t^2}{\left( \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) t - z \right)^3} dH_{2n}(t) \left[ 1 + \frac{1}{N^2} \sum_{j=1}^n q_j \left( \text{Etr} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \mathbf{D}^{-1}(z) \mathbf{T}_{2n} \right) \right] \\
&= \int \frac{t^2}{\left( \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) t - z \right)^3} dH_{2n}(t) \left[ 1 - \frac{1}{N} \sum_{j=1}^n q_j \int \frac{t^2}{\left( \frac{1}{N} \sum_{j=1}^n s_j \psi_j(z) t - z \right)^2} dH_{2n}(t) \right]^{-1}.
\end{aligned}$$

Finally, (3.29), (3.50)-(3.52), and

$$\frac{1}{N} \sum_{j=1}^n q_j = c_n \int \frac{x^2}{(1 + x g_{2n}^0(z))^2} dH_{1n}(x)$$

one has

$$\begin{aligned}
d_{n1}(z) &= -c_n \int \frac{x^2}{(1 + x g_{2n}^0(z))^2} dH_{1n}(x) \int \frac{t^2}{(z g_{1n}^0(z) t + z)^3} dH_{2n}(t) \\
&\times \left[ 1 - c_n \int \frac{x^2}{(1 + x g_{2n}^0(z))^2} dH_{1n}(x) \int \frac{t^2}{(z g_{1n}^0(z) t + z)^2} dH_{2n}(t) \right]^{-1} \\
&- c_n \int \frac{x^3}{(1 + x g_{2n}^0(z))^3} dH_{1n}(x) \int \frac{t}{(z g_{1n}^0(z) t + z)^2} dH_{2n}(t) \int \frac{t^2}{(z g_{1n}^0(z) t + z)^2} dH_{2n}(t) \\
&\times \left[ 1 - c_n \int \frac{x^2}{(1 + x g_{2n}^0(z))^2} dH_{1n}(x) \int \frac{t^2}{(z g_{1n}^0(z) t + z)^2} dH_{2n}(t) \right]^{-1} + o(1).
\end{aligned}$$

and

$$\begin{aligned}
d_{n2}(z) &= -c_n \int \frac{x^2}{(1 + x g_{2n}^0(z))^3} dH_{1n}(x) \int \frac{t^2}{(z g_{1n}^0(z) t + z)^2} dH_{2n}(t) \\
&\times \left[ \int \frac{x}{(1 + x g_{2n}^0(z))^2} dH_{1n}(x) \right]^{-1} \int \frac{x^2}{(1 + x g_{2n}^0(z))^2} dH_{1n}(x) \int \frac{t}{(z + z g_{1n}^0(z) t)^2} dH_{2n}(t)
\end{aligned}$$

$$\times \left[ 1 - c_n \int \frac{x^2}{(1 + xg_{2n}^0(z))^2} dH_{1n}(x) \int \frac{t^2}{(zg_{1n}^0(z)t + z)^2} dH_{2n}(t) \right]^{-1} + o(1).$$

Consequently, from (3.42) and the above two equalities, we conclude that

$$\begin{aligned} M_{n2}(z) &\rightarrow (d_1(z) - d_2(z)) \left\{ 1 - z^{-1} \left[ \int \frac{x}{(1 + xg_2(z))^2} dH_1(x) \right]^{-1} \right. \\ &\quad \times \left. \int \frac{x^2}{(1 + xg_2(z))^2} dH_1(x) \int \frac{t}{(1 + g_1(z)t)^2} dH_2(t) \right\}^{-1}. \end{aligned}$$

#### 4. NON-GAUSSIAN CASE

It has been verified that Lemma 2.3 is true when the entries of the matrix are independent Gaussian variables. This section is to show this conclusion still holds in the general case. The strategy is to compare the characteristic functions of the linear spectral statistics under the normal case and the general case.

We below assume that  $x_{jk}, j = 1, \dots, N, k = 1, \dots, n$  are truncated at  $\delta_n \sqrt{n}$ , centralized and renormalized as in the last section. That is to say,

$$|x_{jl}| \leq \delta_n \sqrt{n}, \quad \text{Ex}_{jl} = 0, \quad \text{Ex}_{jl}^2 = 1, \quad \text{Ex}_{jl}^4 = 3 + o(1).$$

**4.1. From the general case to the Gaussian case.** Denote  $\mathbf{A}_n = \frac{1}{N} \mathbf{T}_{2n}^{1/2} \mathbf{Y}_n \mathbf{T}_{1n} \mathbf{Y}_n' \mathbf{T}_{2n}^{1/2}$  where the entries of  $\mathbf{Y}_n = (y_{jk})$  are independent real Gaussian random variables such that

$$\text{E}y_{jk} = 0, \quad \text{E}y_{jk}^2 = 1, \quad \text{for } j = 1 \dots N, k = 1, \dots, n.$$

Moreover, suppose that  $\mathbf{X}_n$  and  $\mathbf{Y}_n$  be independent random matrices. As in [7] for any  $\theta \in [0, \pi/2]$ , we introduce the following matrices

$$(4.1) \quad \mathbf{W}_n(\theta) = \mathbf{X}_n \sin \theta + \mathbf{Y}_n \cos \theta \quad \text{and} \quad \mathbf{G}_n(\theta) = \frac{1}{N} \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n} \mathbf{W}_n' \mathbf{T}_{2n}^{1/2}$$

where

$$(\mathbf{W}_n(\theta))_{jk} = w_{jk} = x_{jk} \sin \theta + y_{jk} \cos \theta.$$

Furthermore, let

$$(4.2) \quad \begin{aligned} \mathbf{H}_n(t, \theta) &= e^{it\mathbf{G}_n(\theta)}, \quad S(\theta) = \text{tr} f(\mathbf{G}_n(\theta)), \\ S^0(\theta) &= S(\theta) - N \int f(x) dF^{c_n, H_{1n}, H_{2n}}(x), \quad Z_n(x, \theta) = \text{E} e^{ixS^0(\theta)}. \end{aligned}$$

For simplicity, we omit the argument  $\theta$  from the notations of  $\mathbf{W}_n(\theta), \mathbf{G}_n(\theta), \mathbf{H}_n(t, \theta)$  and denote them by  $\mathbf{W}_n, \mathbf{G}_n, \mathbf{H}_n(t)$  respectively.

Note that

$$(4.3) \quad Z_n(x, \pi/2) - Z_n(x, 0) = \int_0^{\pi/2} \frac{\partial Z_n(x, \theta)}{\partial \theta} d\theta.$$

The aim is to prove that  $\frac{\partial Z_n(x, \theta)}{\partial \theta}$  converges to zero uniformly in  $\theta$  over the interval  $[0, \pi/2]$ , which ensures Lemma 2.3.

To this end, let  $f(\lambda)$  be a smooth function with the Fourier transform given by

$$\widehat{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\lambda) e^{-it\lambda} d\lambda.$$

From Lemma A.6, we have

$$\frac{\partial Z_n(x, \theta)}{\partial \theta} = \frac{2xi}{N} \sum_{j=1}^N \sum_{k=1}^n \text{E} w'_{jk} \left[ \mathbf{T}_{2n}^{1/2} \widetilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n} \right]_{jk} e^{ixS^0(\theta)}$$

where

$$w'_{jk} = \frac{dw_{jk}}{d\theta} = x_{jk} \cos \theta - y_{jk} \sin \theta$$

and

$$(4.4) \quad \widetilde{f}(\mathbf{G}_n) = i \int_{-\infty}^{\infty} u \widehat{f}(u) \mathbf{H}_n(u) du.$$

Let  $\mathbf{W}_{njk}(x)$  denote the corresponding matrix  $\mathbf{W}_n$  with  $w_{jk}$  replaced by  $x$ . And let  $\mathbf{G}_{njk}(x) = \frac{1}{N} \mathbf{T}_{2n}^{1/2} \mathbf{W}_{njk}(x) \mathbf{T}_{1n} \mathbf{W}'_{njk}(x) \mathbf{T}_{2n}^{1/2}$ ,

$$\varphi_{jk}(x) = \left[ \mathbf{T}_{2n}^{1/2} \widetilde{f}(\mathbf{G}_{njk}(x)) \mathbf{T}_{2n}^{1/2} \mathbf{W}_{njk}(x) \mathbf{T}_{1n} \right]_{jk} e^{ixS^0(\mathbf{G}_{njk}(x))}.$$

By Taylor's formula, one finds

$$\varphi_{jk}(w_{jk}) = \sum_{l=0}^3 \frac{1}{l!} w_{jk}^l \varphi_{jk}^{(l)}(0) + \frac{1}{4!} w_{jk}^4 \varphi_{jk}^{(4)}(\varrho w_{jk}) \quad \varrho \in (0, 1)$$

which implies that

$$\frac{\partial Z_n(x, \theta)}{\partial \theta} = \frac{2xi}{N} \sum_{l=0}^3 \frac{1}{l!} \sum_{j=1}^N \sum_{k=1}^n \text{E} w'_{jk} w_{jk}^l \text{E} \varphi_{jk}^{(l)}(0) + \frac{2xi}{4!N} \sum_{j=1}^N \sum_{k=1}^n w'_{jk} w_{jk}^4 \varphi_{jk}^{(4)}(\varrho w_{jk}).$$

It is easy to obtain

$$\begin{aligned} \text{E} w'_{jk} w_{jk}^0 &= 0, & \text{E} w'_{jk} w_{jk}^1 &= 0, \\ \text{E} w'_{jk} w_{jk}^2 &= \text{E} w_{jk}^3 \sin^2 \theta \cos \theta, & \text{E} w'_{jk} w_{jk}^3 &= o(1) \sin^3 \theta \cos \theta. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{\partial Z_n(x, \theta)}{\partial \theta} &= \frac{xi}{N} \sum_{j=1}^N \sum_{k=1}^n \text{E} w_{jk}^3 \sin^2 \theta \cos \theta \text{E} \varphi_{jk}^{(2)}(0) + \frac{xi}{12N} \sum_{j=1}^N \sum_{k=1}^n \text{E} w'_{jk} w_{jk}^4 \varphi_{jk}^{(4)}(\varrho w_{jk}) \\ &\triangleq \mathcal{I}_1 + \mathcal{I}_2. \end{aligned}$$

We below analyze  $\mathcal{I}_1$  and  $\mathcal{I}_2$  term by term.



4.1.1. *The second derivative.* We first consider  $\mathcal{I}_1$ . A direct calculation yields that

$$\begin{aligned}\varphi_{jk}^{(2)}(w_{jk}) &= \left[ \mathbf{T}_{2n}^{1/2} \frac{\partial^2 \tilde{f}(\mathbf{G}_n)}{\partial w_{jk}^2} \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n} \right]_{jk} e^{ixS_0(\theta)} + 2 \left[ \mathbf{T}_{2n}^{1/2} \frac{\partial \tilde{f}(\mathbf{G}_n)}{\partial w_{jk}} \mathbf{T}_{2n}^{1/2} \right]_{jj} [\mathbf{T}_{1n}]_{kk} e^{ixS_0(\theta)} \\ &\quad + \frac{6xi}{N} \left[ \mathbf{T}_{2n}^{1/2} \frac{\partial \tilde{f}(\mathbf{G}_n)}{\partial w_{jk}} \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n} \right]_{jk} \left[ \mathbf{T}_{2n}^{1/2} \tilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n} \right]_{jk} e^{ixS_0(\theta)} \\ &\quad + \frac{6xi}{N} \left[ \mathbf{T}_{2n}^{1/2} \tilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n} \right]_{jk} \left[ \mathbf{T}_{2n}^{1/2} \tilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2} \right]_{jj} [\mathbf{T}_{1n}]_{kk} e^{ixS_0(\theta)} \\ &\quad - \frac{4x^2}{N^2} \left[ \mathbf{T}_{2n}^{1/2} \tilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n} \right]_{jk}^3 e^{ixS_0(\theta)} \\ &\triangleq \mathcal{J}_{jk}^1 + \mathcal{J}_{jk}^2 + \mathcal{J}_{jk}^3 + \mathcal{J}_{jk}^4 + \mathcal{J}_{jk}^5.\end{aligned}$$

Using Lemma A.6, one finds

$$\begin{aligned}\mathcal{J}_{jk}^1 &= -\frac{2}{N} \int_{-\infty}^{\infty} u \widehat{f}(u) [\mathbf{T}_{1n}]_{kk} [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2}]_{jj} * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk}(u) e^{ixS_0(\theta)} du \\ &\quad - \frac{6i}{N^2} \int_{-\infty}^{\infty} u \widehat{f}(u) [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2}]_{jj} * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} \\ &\quad * [\mathbf{T}_{1n} \mathbf{W}_n' \mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{kk}(u) e^{ixS_0(\theta)} du \\ &\quad - \frac{2i}{N^2} \int_{-\infty}^{\infty} u \widehat{f}(u) [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} \\ &\quad * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk}(u) e^{ixS_0(\theta)} du.\end{aligned}$$

It is straightforward to check that the moments of  $\|\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2}\|$ ,  $\frac{1}{\sqrt{N}} \|\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}\|$  and

$$(4.5) \quad \frac{1}{N} \|\mathbf{T}_{1n} \mathbf{W}_n' \mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}\|$$

are bounded. Applying Lemma A.7, we obtain

$$\left| \frac{1}{N} \sum_{j=1}^N \sum_{k=1}^n \mathbb{E} \mathcal{J}_{jk}^1 \right| \leq \frac{C}{N^{1/4}} \int_{-\infty}^{\infty} (|u|^2 + |u|^3) |\widehat{f}(u)| du \leq CN^{-1/4}.$$

By the same argument, we get

$$\left| \frac{1}{N} \sum_{j=1}^N \sum_{k=1}^n \mathbb{E} (\mathcal{J}_{jk}^2 + \mathcal{J}_{jk}^3 + \mathcal{J}_{jk}^4 + \mathcal{J}_{jk}^5) \right| \leq CN^{-1/4}.$$

Hence,

$$|\mathcal{I}_1| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

4.1.2. *The remainder term.* It is straightforward to check that

$$E w'_{jk} w_{jk}^4 \leq C \delta_n \sqrt{n}.$$

Let  $w$  be a random variable which has the same first, second and fourth moments as  $w_{jk}$ . We estimate  $E \sup_w \varphi_{jk}^{(4)}(w)$ . A direct but tedious computation yields

$$\begin{aligned} & \varphi_{jk}^{(4)}(w_{jk}) \\ = & [\mathbf{T}_{2n}^{1/2} \frac{\partial^4 \tilde{f}(\mathbf{G}_n)}{\partial w_{jk}^4} \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} e^{ixS_0(\theta)} + 4[\mathbf{T}_{1n}]_{kk} [\mathbf{T}_{2n}^{1/2} \frac{\partial^3 \tilde{f}(\mathbf{G}_n)}{\partial w_{jk}^3} \mathbf{T}_{2n}^{1/2}]_{jj} e^{ixS_0(\theta)} \\ & + \frac{10xi}{N} [\mathbf{T}_{2n}^{1/2} \frac{\partial^3 \tilde{f}(\mathbf{G}_n)}{\partial w_{jk}^3} \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} [\mathbf{T}_{2n}^{1/2} \tilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} e^{ixS_0(\theta)} \\ & + \frac{30xi}{N} [\mathbf{T}_{1n}]_{kk} [\mathbf{T}_{2n}^{1/2} \frac{\partial^2 \tilde{f}(\mathbf{G}_n)}{\partial w_{jk}^2} \mathbf{T}_{2n}^{1/2}]_{jj} [\mathbf{T}_{2n}^{1/2} \tilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} e^{ixS_0(\theta)} \\ & + \frac{20xi}{N} [\mathbf{T}_{2n}^{1/2} \frac{\partial^2 \tilde{f}(\mathbf{G}_n)}{\partial w_{jk}^2} \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} [\mathbf{T}_{2n}^{1/2} \frac{\partial \tilde{f}(\mathbf{G}_n)}{\partial w_{jk}} \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} e^{ixS_0(\theta)} \\ & + \frac{20xi}{N} [\mathbf{T}_{1n}]_{kk} [\mathbf{T}_{2n}^{1/2} \tilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2}]_{jj} [\mathbf{T}_{2n}^{1/2} \frac{\partial^2 \tilde{f}(\mathbf{G}_n)}{\partial w_{jk}^2} \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} e^{ixS_0(\theta)} \\ & + \frac{40xi}{N} [\mathbf{T}_{1n}]_{kk} [\mathbf{T}_{2n}^{1/2} \frac{\partial \tilde{f}(\mathbf{G}_n)}{\partial w_{jk}} \mathbf{T}_{2n}^{1/2}]_{jj} [\mathbf{T}_{2n}^{1/2} \frac{\partial \tilde{f}(\mathbf{G}_n)}{\partial w_{jk}} \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} e^{ixS_0(\theta)} \\ & + \frac{40xi}{N} [\mathbf{T}_{1n}]_{kk}^2 [\mathbf{T}_{2n}^{1/2} \tilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2}]_{jj} [\mathbf{T}_{2n}^{1/2} \frac{\partial \tilde{f}(\mathbf{G}_n)}{\partial w_{jk}} \mathbf{T}_{2n}^{1/2}]_{jj} e^{ixS_0(\theta)} \\ & - \frac{40x^2}{N^2} [\mathbf{T}_{2n}^{1/2} \frac{\partial^2 \tilde{f}(\mathbf{G}_n)}{\partial w_{jk}^2} \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} [\mathbf{T}_{2n}^{1/2} \tilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk}^2 e^{ixS_0(\theta)} \\ & - \frac{60x^2}{N^2} [\mathbf{T}_{2n}^{1/2} \frac{\partial \tilde{f}(\mathbf{G}_n)}{\partial w_{jk}} \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk}^2 [\mathbf{T}_{2n}^{1/2} \tilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} e^{ixS_0(\theta)} \\ & - \frac{80x^2}{N^2} [\mathbf{T}_{1n}]_{kk} [\mathbf{T}_{2n}^{1/2} \frac{\partial \tilde{f}(\mathbf{G}_n)}{\partial w_{jk}} \mathbf{T}_{2n}^{1/2}]_{jj} [\mathbf{T}_{2n}^{1/2} \tilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk}^2 e^{ixS_0(\theta)} \\ & - \frac{60x^2}{N^2} [\mathbf{T}_{1n}]_{kk}^2 [\mathbf{T}_{2n}^{1/2} \tilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2}]_{jj}^2 [\mathbf{T}_{2n}^{1/2} \tilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} e^{ixS_0(\theta)} \\ & - \frac{80x^3 i}{N^3} [\mathbf{T}_{2n}^{1/2} \frac{\partial \tilde{f}(\mathbf{G}_n)}{\partial w_{jk}} \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} [\mathbf{T}_{2n}^{1/2} \tilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk}^3 e^{ixS_0(\theta)} \\ & - \frac{80x^3 i}{N^3} [\mathbf{T}_{1n}]_{kk} [\mathbf{T}_{2n}^{1/2} \tilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2}]_{jj} [\mathbf{T}_{2n}^{1/2} \tilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk}^3 e^{ixS_0(\theta)} \end{aligned}$$

$$\begin{aligned}
& - \frac{120x^2}{N^2} [\mathbf{T}_{1n}]_{kk} [\mathbf{T}_{2n}^{1/2} \frac{\partial \widetilde{f}(\mathbf{G}_n)}{\partial w_{jk}} \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} [\mathbf{T}_{2n}^{1/2} \widetilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} \\
& \times [\mathbf{T}_{2n}^{1/2} \widetilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2}]_{jj} e^{ixS_0(\theta)} + \frac{16x^4}{N^4} [\mathbf{T}_{2n}^{1/2} \widetilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk}^5 e^{ixS_0(\theta)}.
\end{aligned}$$

We only estimate the first term  $[\mathbf{T}_{2n}^{1/2} \frac{\partial^4 \widetilde{f}(\mathbf{G}_n)}{\partial w_{jk}^4} \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} e^{ixS_0(\theta)}$  and the others are similar. Simple calculations imply that

$$\begin{aligned}
& [\mathbf{T}_{2n}^{1/2} \frac{\partial^4 \widetilde{f}(\mathbf{G}_n)}{\partial w_{jk}^4} \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} e^{ixS_0(\theta)} \\
& = - \frac{24i}{N^2} \int_{-\infty}^{\infty} u \widehat{f}(u) [\mathbf{T}_{1n}]_{kk}^2 [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2}]_{jj} * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2}]_{jj} \\
& \quad * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk}(u) e^{ixS_0(\theta)} du \\
& \quad + \frac{144}{N^3} \int_{-\infty}^{\infty} u \widehat{f}(u) [\mathbf{T}_{1n}]_{kk} [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2}]_{jj} * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} \\
& \quad * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk}(u) e^{ixS_0(\theta)} du \\
& \quad + \frac{144}{N^3} \int_{-\infty}^{\infty} u \widehat{f}(u) [\mathbf{T}_{1n}]_{kk} [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2}]_{jj} * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2}]_{jj} \\
& \quad * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} * [\mathbf{T}_{1n} \mathbf{W}_n' \mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{kk}(u) e^{ixS_0(\theta)} du \\
& \quad + \frac{240i}{N^4} \int_{-\infty}^{\infty} u \widehat{f}(u) [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2}]_{jj} * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} \\
& \quad * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} * [\mathbf{T}_{1n} \mathbf{W}_n' \mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{kk}(u) e^{ixS_0(\theta)} du \\
& \quad + \frac{120i}{N^4} \int_{-\infty}^{\infty} u \widehat{f}(u) [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2}]_{jj} * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2}]_{jj} * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} \\
& \quad * [\mathbf{T}_{1n} \mathbf{W}_n' \mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{kk} * [\mathbf{T}_{1n} \mathbf{W}_n' \mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{kk}(u) e^{ixS_0(\theta)} du \\
& \quad + \frac{24i}{N^4} \int_{-\infty}^{\infty} u \widehat{f}(u) [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} \\
& \quad * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk}(u) e^{ixS_0(\theta)} du \\
& \triangleq Q_{jk}^1 + Q_{jk}^2 + Q_{jk}^3 + Q_{jk}^4 + Q_{jk}^5 + Q_{jk}^6.
\end{aligned}$$

From (4.5), we deduce that

$$\begin{aligned}
\left| \frac{1}{N} \sum_{j,k} \mathbb{E} Q_{jk}^1 \right| & \leq \frac{C}{N} \int_{-\infty}^{\infty} |u|^3 \widehat{f}(u) \mathbb{E} \left\| \mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \right\|^2 \left\| \mathbf{T}_{2n}^{1/2} \mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n} \right\| du \\
& \leq \frac{C}{\sqrt{N}} \int_{-\infty}^{\infty} |u|^3 \widehat{f}(u) du = O(N^{-1/2}).
\end{aligned}$$

Applying the same arguments as above one can conclude that

$$\begin{aligned} & \left| \frac{1}{N} \sum_{j,k} \mathbb{E} (Q_{jk}^2 + Q_{jk}^3 + Q_{jk}^4 + Q_{jk}^5 + Q_{jk}^6) \right| \\ & \leq \frac{C}{\sqrt{N}} \int_{-\infty}^{\infty} (|u|^4 + |u|^5) \widehat{f}(u) du = O(N^{-1/2}). \end{aligned}$$

It follows that

$$|\mathcal{I}_2| \leq C\delta_n \rightarrow 0.$$

This fact finishes the proof of Lemma 2.3.

#### APPENDIX A.

This section is to prove some lemmas which are used in the proof of Lemma 2.3.

**Lemma A.1.** *Under the conditions of Theorem 2.1, we have for  $z \in C_u$  and  $p \geq 1$*

$$\mathbb{E}|\beta_k(z)|^p \leq C, \mathbb{E}|\widetilde{\beta}_k(z)| \leq C, |b_k(z)| \leq C, |\psi_k(z)| \leq C$$

where  $\beta_k(z), \widetilde{\beta}_k(z), b_k(z), \psi_k(z)$  are defined in (3.5) and (3.6).

*Proof.* We only prove  $\mathbb{E}|\beta_k(z)|^p \leq C$  and the others are similar. Note that

$$N^{-1} |\mathbf{y}'_k \mathbf{D}_k^{-1}(z) \mathbf{y}_k| \leq \frac{\|\mathbf{y}_k\|^2}{Nv_0}$$

which gives

$$(A.1) \quad |\beta_k(z)| \leq \frac{1}{1 - \frac{|s_k| \|\mathbf{y}_k\|^2}{Nv_0}} < 2 \quad \text{if } \frac{|s_k| \|\mathbf{y}_k\|^2}{Nv_0} \leq 1/2$$

where  $\|\mathbf{y}_k\|^2 = \sum_{j=1}^N y_{jk}^2$ . Denoting by  $\mathbf{O}\mathbf{\Lambda}\mathbf{O}^*$  the spectral decomposition of  $\mathbf{B}_{nk}$  and  $\mathbf{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_N)$ , we obtain

$$\begin{aligned} |\beta_k(z)| & \leq \frac{1}{\left| \Im \left( N^{-1} s_k \mathbf{y}'_k \mathbf{D}_k^{-1}(z) \mathbf{y}_k \right) \right|} = \frac{1}{\left| N^{-1} s_k v_0 \sum_{j=1}^N \frac{(\mathbf{O}^* \mathbf{y}_k \mathbf{y}'_k \mathbf{O})_{jj}}{(\lambda_j - u)^2 + v_0^2} \right|} \\ (A.2) \quad & \leq \frac{2 \max_{1 \leq j \leq N} \lambda_j^2 + 2|z|^2}{N^{-1} v_0 \|\mathbf{y}_k\|^2 |s_k|} \leq \frac{4 \max_{1 \leq j \leq N} \lambda_j^2 + 4|z|^2}{v_0^2} \quad \text{if } \frac{|s_k| \|\mathbf{y}_k\|^2}{Nv_0} > 1/2. \end{aligned}$$

Combining (A.1) with (A.2), we have

$$|\beta_k(z)| \leq \frac{4 \max_{1 \leq j \leq N} \lambda_j^2 + 4|z|^2}{v_0^2} + 2 \leq \frac{4 \max_{1 \leq j \leq N} \lambda_j^2 + 6|z|^2}{v_0^2}.$$

Moreover,

$$\max_{1 \leq j \leq N} \lambda_j^2 \leq \text{tr} \mathbf{B}_{nk}^2 \leq \frac{\tau^4}{N} (\mathbf{x}'_k \mathbf{x}_k)^2 \leq \tau^4 \delta_n^4 N n^2 \leq C N^3.$$

It follows that

$$(A.3) \quad |\beta_k(z)| \leq \begin{cases} \frac{4\eta_l^2 + 4\eta_r^2 + 6|z|^2}{v_0^2} \leq C, & \text{if } \eta_l \leq \lambda_{\min} \leq \lambda_{\max} \leq \eta_r, \\ CN^3, & \text{otherwise.} \end{cases}$$

Using (3.1), (3.2) and (A.3), we have for suitably large  $l$

$$\begin{aligned} E|\beta_k(z)|^p &\leq E|\beta_k(z)|^p I(\eta_l \leq \lambda_{\min} \leq \lambda_{\max} \leq \eta_r) + E|\beta_k(z)|^p I(\lambda_{\min} < \eta_l \text{ or } \lambda_{\max} > \eta_r) \\ &\leq C + CN^3 P(\lambda_{\min} < \eta_l \text{ or } \lambda_{\max} > \eta_r) \\ &\leq C + CN^3 n^{-l} \leq C. \end{aligned}$$

This completes the proof of this lemma.  $\square$

**Lemma A.2.** *Under the conditions of Theorem 2.1, we have*

$$\sup_{z \in C_n} |Eg_{2n}(z) - g_{2n}^0(z)| \rightarrow 0.$$

*Proof.* Similarly to the proof of (4.1) in [1], one can check that

$$\sup_{z \in C_n} |E\underline{m}_n(z) - \underline{m}(z)| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

By Lemma B.1, (3.38) and (3.41), we have

$$\sup_{z \in C_n} \left| \frac{1}{n} \sum_{j=1}^n (E\beta_j(z) - \psi_j(z)) \right| \leq C |E\beta_j(z) - \psi_j(z)| \leq \frac{C}{N} \rightarrow 0.$$

Note that

$$\begin{aligned} |Eg_{2n}(z) - g_{2n}^0(z)| &\left| \frac{1}{n} \sum_{j=1}^n \frac{s_j}{(1 + s_j Eg_{2n}(z))(1 + s_j g_{2n}^0(z))} \right| \\ &= \left| \frac{1}{n} \sum_{j=1}^n \left( \psi_j(z) - \frac{1}{1 + s_j g_{2n}^0(z)} \right) \right| \\ &\leq \left| \frac{1}{n} \sum_{j=1}^n (E\beta_j(z) - \psi_j(z)) \right| + |z| |E\underline{m}_n(z) - \underline{m}(z)| + |z| |\underline{m}_n^0(z) - \underline{m}(z)| \rightarrow 0. \end{aligned}$$

Consequently, it suffices to prove that  $\left| \frac{1}{n} \sum_{j=1}^n \frac{s_j}{(1 + s_j Eg_{2n}^0(z))(1 + s_j g_{2n}^0(z))} \right|$  has a lower bound.

We prove it by contradiction. By (1.6), one has

$$(A.4) \quad (z\underline{m}(z))' = g_2'(z) \int \frac{x}{(1 + g_2(z)x)^2} dH_1.$$

Suppose that there exists a sequence  $\{z_h \in C_n\}$  such that  $z_h \rightarrow z_0$  and

$$\int \frac{x}{(1 + g_2(z_h)x)^2} dH_1(x) \rightarrow 0.$$

From (A.4) and the continuity of  $g_2(z)$ , it follows that

$$(A.5) \quad (z_h \underline{m}(z_h))' \rightarrow (z_0 \underline{m}(z_0))' = 0.$$

Then from the equation  $m(z_0) + z_0 m'(z_0) = 0$ , we obtain another solution of  $m(z)$  which has nothing to do with  $H_1(z)$  and  $H_2(z)$ . However, this contradicts to the fact that  $m(z)$  is a unique solution of (1.3). Hence, we get

$$\left| \int \frac{x}{(1 + g_2(z)x)^2} dH_1(x) \right| > 0.$$

By continuity and convergence of  $Eg_{2n}(z)$ , we see that for all large  $n$

$$\left| \frac{1}{n} \sum_{j=1}^n \frac{s_j}{(1 + s_j Eg_{2n}(z))(1 + s_j g_{2n}^0(z))} \right| > 0.$$

Therefore, we conclude that

$$\sup_{z \in C_n} |Eg_{2n}(z) - g_{2n}^0(z)| \rightarrow 0.$$

□

**Lemma A.3.** *For  $z \in C_n$ , we have for any positive  $p \geq 1$*

$$E|\beta_j(z)|^p \leq C^p$$

where  $\beta_j(z)$  is defined in (3.5).

*Proof.* By formula (3.7), we get

$$\mathbf{D}^{-1}(z) - \mathbf{D}_j^{-1}(z) = -\frac{1}{N} s_j \beta_j(z) \mathbf{D}_j^{-1} \mathbf{y}_j \mathbf{y}_j' \mathbf{D}_j^{-1}.$$

This yields

$$\begin{aligned} \frac{1}{N} s_j \mathbf{y}_j' \mathbf{D}^{-1}(z) \mathbf{y}_j &= \frac{1}{N} s_j \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{y}_j \left( 1 - \frac{1}{N} s_j \beta_j(z) \mathbf{y}_j' \mathbf{D}_j^{-1} \mathbf{y}_j \right) \\ &= \frac{1}{N} s_j \beta_j(z) \mathbf{y}_j' \mathbf{D}_j^{-1}(z) \mathbf{y}_j = 1 - \beta_j(z). \end{aligned}$$

If  $\eta_l < \lambda_{\min}^{\mathbf{B}_n} \leq \lambda_{\max}^{\mathbf{B}_n} < \eta_r$  and  $\eta_l < \lambda_{\min}^{\mathbf{B}_{nj}} \leq \lambda_{\max}^{\mathbf{B}_{nj}} < \eta_r$ , then we have

$$\begin{aligned} \left| \frac{1}{N} s_j \mathbf{y}_j' \mathbf{y}_j \right| &= \left| \max_{\|\mathbf{f}\|=1} \mathbf{f}' (\mathbf{B}_n - \mathbf{B}_{nj}) \mathbf{f} \right| \leq \left| \max_{\|\mathbf{f}\|=1} \mathbf{f}' \mathbf{B}_n \mathbf{f} \right| + \left| \min_{\|\mathbf{f}\|=1} \mathbf{f}' \mathbf{B}_{nj} \mathbf{f} \right| \\ &= \left| \lambda_{\max}^{\mathbf{B}_n} \right| + \left| \lambda_{\min}^{\mathbf{B}_{nj}} \right| \leq 2(|\eta_r| + |\eta_l|). \end{aligned}$$

Otherwise,

$$\left| \frac{1}{N} s_j \mathbf{y}_j' \mathbf{y}_j \right| \leq \frac{\tau^2}{N} \sum_{k=1}^N |x_{jk}|^2 \leq \tau^2 \delta_n^2 n \leq n.$$

Therefore, one has

$$|\beta_j(z)| = |1 - \frac{1}{N} s_j \mathbf{y}'_j \mathbf{D}^{-1}(z) \mathbf{y}_j| \leq 1 + 2(|\eta_r| + |\eta_l|) \max\{\frac{1}{x_r - \eta_r}, \frac{1}{\eta_l - x_l}, \frac{1}{v_0}\} \\ + n^{2+\alpha} I(\lambda_{\min}^{\mathbf{B}_n} < \eta_l \text{ or } \lambda_{\max}^{\mathbf{B}_n} < \eta_r \text{ or } \lambda_{\min}^{\mathbf{B}_{nj}} > \eta_l \text{ or } \lambda_{\max}^{\mathbf{B}_{nj}} < \eta_r).$$

By (3.1) and (3.2), we have for any positive  $p \geq 1$  and  $l > 3$

$$\mathbb{E}|\beta_j(z)|^p \leq C_1^p + C_2^p n^{-l} \leq C^p.$$

□

**Lemma A.4.** *Let  $1 \leq j \leq N, 1 \leq k \leq n$ . Recall the definition of  $\mathbf{G}_n$  in (4.1). Then, for any  $1 \leq a, b \leq N$ , we have*

$$\frac{\partial g_{ab}}{\partial w_{jk}} = \left[ \frac{\partial \mathbf{G}_n}{\partial w_{jk}} \right]_{ab} = \frac{1}{N} [\mathbf{T}_{2n}^{1/2}]_{aj} [\mathbf{T}_{1n} \mathbf{W}'_n \mathbf{T}_{2n}^{1/2}]_{kb} + \frac{1}{N} [\mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{ak} [\mathbf{T}_{2n}^{1/2}]_{jb}.$$

*Proof.* It is obvious that

$$\frac{\partial \mathbf{G}_n}{\partial w_{jk}} = \frac{1}{N} \mathbf{T}_{2n}^{1/2} \frac{\partial \mathbf{W}_n}{\partial w_{jk}} \mathbf{T}_{1n} \mathbf{W}'_n \mathbf{T}_{2n}^{1/2} + \frac{1}{N} \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n} \frac{\partial \mathbf{W}'_n}{\partial w_{jk}} \mathbf{T}_{2n}^{1/2} \\ = \frac{1}{N} \mathbf{T}_{2n}^{1/2} \mathbf{e}_j \mathbf{e}'_k \mathbf{T}_{1n} \mathbf{W}'_n \mathbf{T}_{2n}^{1/2} + \frac{1}{N} \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n} \mathbf{e}_k \mathbf{e}'_j \mathbf{T}_{2n}^{1/2}.$$

This yields

$$\left[ \frac{\partial \mathbf{G}_n}{\partial w_{jk}} \right]_{ab} = \frac{1}{N} [\mathbf{T}_{2n}^{1/2}]_{aj} [\mathbf{T}_{1n} \mathbf{W}'_n \mathbf{T}_{2n}^{1/2}]_{kb} + \frac{1}{N} [\mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{ak} [\mathbf{T}_{2n}^{1/2}]_{jb}.$$

□

**Lemma A.5.** *Let  $1 \leq j \leq N, 1 \leq k \leq n$ . Recall the definition of  $\mathbf{H}_n(t)$  in (4.2). Then for any  $1 \leq d, l \leq N$*

$$\frac{\partial h_{dl}}{\partial w_{jk}} = \frac{i}{N} [\mathbf{H}_n \mathbf{T}_{2n}^{1/2}]_{dj} * [\mathbf{T}_{1n} \mathbf{W}'_n \mathbf{T}_{2n}^{1/2} \mathbf{H}_n]_{kl}(t) + \frac{i}{N} [\mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{dk} * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n]_{jl}(t)$$

where  $h_{dl} = (\mathbf{H}_n(t))_{dl}$  and  $f * g(t) = \int_0^t f(s)g(t-s)ds$ .

*Proof.* Applying Lemma A.4 and Lemma B.8, we get

$$\frac{\partial \mathbf{H}_n(t)}{\partial w_{jk}} = \sum_{a,b=1}^N \frac{\partial \mathbf{H}_n(t)}{\partial g_{ab}} \frac{\partial g_{ab}}{\partial w_{jk}} = \frac{i}{N} \sum_{a,b=1}^N \int_0^t e^{is\mathbf{G}_n} \mathbf{e}_j \mathbf{e}'_k e^{(1-s)\mathbf{G}_n} ds \\ \times \left\{ [\mathbf{T}_{2n}^{1/2}]_{aj} [\mathbf{T}_{1n} \mathbf{W}'_n \mathbf{T}_{2n}^{1/2}]_{kb} + [\mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{ak} [\mathbf{T}_{2n}^{1/2}]_{jb} \right\}.$$

Hence, one has

$$\frac{\partial h_{dl}}{\partial w_{jk}} = \frac{i}{N} \sum_{a,b=1}^N h_{da} * h_{bl}(t) \left\{ [\mathbf{T}_{2n}^{1/2}]_{aj} [\mathbf{T}_{1n} \mathbf{W}'_n \mathbf{T}_{2n}^{1/2}]_{kb} + [\mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{ak} [\mathbf{T}_{2n}^{1/2}]_{jb} \right\}$$

$$= \frac{i}{N} [\mathbf{H}_n \mathbf{T}_{2n}^{1/2}]_{dj} * [\mathbf{T}_{1n} \mathbf{W}'_n \mathbf{T}_{2n}^{1/2} \mathbf{H}_n]_{kl}(t) + \frac{i}{N} [\mathbf{H}_n \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{dk} * [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n]_{jl}(t).$$

□

**Lemma A.6.** *Let  $1 \leq j \leq N, 1 \leq k \leq n$ . Recall the definitions of  $S(\theta)$  in (4.2) and  $\tilde{f}(\mathbf{G}_n)$  in (4.4). Then for any  $1 \leq d, l \leq N$*

$$\frac{\partial S(\theta)}{\partial w_{jk}} = \frac{2}{N} [\mathbf{T}_{2n}^{1/2} \tilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk}.$$

*Proof.* By the inverse Fourier transform, we obtain

$$\frac{\partial S}{\partial w_{jk}} = \int_{-\infty}^{\infty} \widehat{f}(u) \text{tr} \frac{\partial \mathbf{H}_n(u)}{\partial w_{jk}} du.$$

It follows from Lemma A.5 that

$$\begin{aligned} \frac{\partial S}{\partial w_{jk}} &= \frac{2i}{N} \int_{-\infty}^{\infty} \widehat{f}(u) \sum_{d=1}^N [\mathbf{H}_n \mathbf{T}_{2n}^{1/2}]_{dj} * [\mathbf{T}_{1n} \mathbf{W}'_n \mathbf{T}_{2n}^{1/2} \mathbf{H}_n]_{kd}(u) du \\ &= \frac{2i}{N} \int_{-\infty}^{\infty} u \widehat{f}(u) [\mathbf{T}_{2n}^{1/2} \mathbf{H}_n(u) \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk} du \\ &= \frac{2}{N} [\mathbf{T}_{2n}^{1/2} \tilde{f}(\mathbf{G}_n) \mathbf{T}_{2n}^{1/2} \mathbf{W}_n \mathbf{T}_{1n}]_{jk}. \end{aligned}$$

□

**Lemma A.7.** *Suppose  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$  are  $p \times n$  random matrices and their moments of the spectral norms are bounded. Then we get*

$$(A.6) \quad \left| \sum_{j,k} \mathbf{E} \mathbf{A}_{jj} \mathbf{B}_{jk} \mathbf{C}_{kk} \right| \leq C n^{5/4}$$

$$(A.7) \quad \left| \sum_{j,k} \mathbf{E} \mathbf{A}_{jk} \mathbf{B}_{jk} \mathbf{C}_{jk} \right| \leq C n^{5/4}.$$

*Proof.* Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \left| \sum_{j,k} \mathbf{E} \mathbf{A}_{jj} \mathbf{B}_{jk} \mathbf{C}_{kk} \right| &\leq \left( \sum_j \mathbf{E} |\mathbf{A}_{jj}|^2 \right)^{1/2} \left( \sum_j \mathbf{E} \left| \sum_k \mathbf{B}_{jk} \mathbf{C}_{kk} \right|^2 \right)^{1/2} \\ &\leq C \sqrt{n} \left( \sum_{k_1, k_2} \mathbf{E} (\mathbf{B}' \mathbf{B})_{k_1 k_2} \mathbf{C}_{k_1 k_1} \mathbf{C}'_{k_2 k_2} \right)^{1/2} \\ &\leq C \sqrt{n} \left( \sum_{k_1, k_2} \mathbf{E} (\mathbf{B}' \mathbf{B})_{k_1 k_2}^2 \right)^{1/4} \left( \sum_{k_1, k_2} \mathbf{E} |\mathbf{C}_{k_1 k_1}|^2 |\mathbf{C}'_{k_2 k_2}|^2 \right)^{1/4} \\ &\leq C n^{5/4} \end{aligned}$$



and

$$\begin{aligned}
\left| \sum_{j,k} \mathbf{E} \mathbf{A}_{jk} \mathbf{B}_{jk} \mathbf{C}_{jk} \right| &\leq \left( \sum_{j,k} \mathbf{E} |\mathbf{A}_{jk}| |\mathbf{B}_{jk}| \right)^{1/2} \left( \sum_{j,k} \mathbf{E} |\mathbf{A}_{jk}| |\mathbf{B}_{jk}| |\mathbf{C}_{jk}|^2 \right)^{1/2} \\
&\leq \left( \sum_{j,k} \mathbf{E} |\mathbf{A}_{jk}| |\mathbf{B}_{jk}| \right)^{3/4} \left( \sum_{j,k} \mathbf{E} |\mathbf{A}_{jk}| |\mathbf{B}_{jk}| |\mathbf{C}_{jk}|^4 \right)^{1/4} \\
&\leq C \sqrt{n} \left( \sum_{j,k} \mathbf{E} |\mathbf{A}_{jk}|^2 \right)^{3/8} \left( \sum_{j,k} \mathbf{E} |\mathbf{B}_{jk}|^2 \right)^{3/8} \\
&\leq C n^{5/4}.
\end{aligned}$$

□

## APPENDIX B.

In this section, we list several technical facts that will be often used in the paper.

**Lemma B.1** (Lemma B.26 in [1]). *Let  $\mathbf{A} = (a_{jk})$  be an  $n \times n$  nonrandom matrix and  $\mathbf{X} = (x_1, \dots, x_n)'$  be a random vector of independent entries. Assume that  $\mathbf{E} x_j = 0$ ,  $\mathbf{E} |x_j|^2 = 1$  and  $\mathbf{E} |x_j|^l \leq \nu_l$ . Then for  $p \geq 1$ ,*

$$\mathbf{E} |\mathbf{X}' \mathbf{A} \mathbf{X} - \text{tr} \mathbf{A}|^p \leq C_p \left[ (\nu_4 \text{tr} \mathbf{A} \mathbf{A}')^{p/2} + \nu_{2p} \text{tr} (\mathbf{A} \mathbf{A}')^{p/2} \right]$$

where  $C_p$  is a constant depending on  $p$  only.

**Lemma B.2** (Lemma 2.4 in [1]). *Suppose for each  $n$   $Y_{n1}, Y_{n2}, \dots, Y_{nr_n}$  is a real martingale difference sequence with respect to the increasing  $\sigma$ -field  $\{\mathcal{F}_{nj}\}$  having second moments. If as  $n \rightarrow \infty$ ,*

$$(i) \quad \sum_{j=1}^{r_n} \mathbf{E}(Y_{nj}^2 | \mathcal{F}_{n,j-1}) \xrightarrow{i.p.} \sigma^2,$$

where  $\sigma^2$  is a positive constant, and for each  $\varepsilon \geq 0$ ,

$$(ii) \quad \sum_{j=1}^{r_n} \mathbf{E}(Y_{nj}^2 I(|Y_{nj}| \geq \varepsilon)) \rightarrow 0,$$

then

$$\sum_{j=1}^{r_n} Y_{nj} \xrightarrow{D} N(0, \sigma^2).$$

**Lemma B.3** (Lemma 2.6 in [19]). *Let  $z \in \mathbb{C}^+$  with  $\nu = \Im z$ ,  $\mathbf{A}$  and  $\mathbf{B}$   $N \times N$  with  $\mathbf{B}$  Hermitian,  $\tau \in \mathbb{R}$ , and  $\mathbf{q} \in \mathbb{C}^N$ . Then*

$$\left| \text{tr} \left[ \left( (\mathbf{B} - z\mathbf{I})^{-1} - (\mathbf{B} + \tau \mathbf{q} \mathbf{q}^* - z\mathbf{I})^{-1} \right) \mathbf{A} \right] \right| \leq \frac{\|\mathbf{A}\|}{\nu}.$$

**Lemma B.4** (Abel lemma). *Suppose  $\{f_k\}$  and  $\{r_k\}$  are two sequences. Then, we have*

$$\sum_{k=1}^n f_k(r_{k+1} - r_k) = f_{n+1}r_{n+1} - f_1r_1 - \sum_{k=1}^n r_{k+1}(f_{k+1} - f_k).$$

**Lemma B.5** (inequality (4.8) in [1]). *Let  $\mathbf{M}$  be  $N \times N$  nonrandom matrix, we find for  $j \in \{1, 2, \dots, n\}$*

$$\mathbb{E} |\text{tr} \mathbf{D}_j^{-1} \mathbf{M} - \mathbb{E} \text{tr} \mathbf{D}_j^{-1} \mathbf{M}|^2 \leq C \|\mathbf{M}\|^2$$

**Lemma B.6** (Theorem A.37 in [2]). *If  $\mathbf{A}$  and  $\mathbf{B}$  are two  $n \times p$  matrices and  $\lambda_k, \delta_k, k = 1, 2, \dots, n$  denote their singular values. If the singular values are arranged in descending order, then we have*

$$\sum_{k=1}^v |\lambda_k - \delta_k|^2 \leq \text{tr} [(\mathbf{A} - \mathbf{B})(\mathbf{A} - \mathbf{B})^*]$$

where  $v = \min\{p, n\}$ .

**Lemma B.7.** *For rectangular matrices  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ , we have*

$$|\text{tr}(\mathbf{ABCD})| \leq \|\mathbf{A}\| \|\mathbf{C}\| (\text{tr} \mathbf{BB}^*)^{1/2} (\text{tr} \mathbf{DD}^*)^{1/2}.$$

**Lemma B.8** (Duhamel formula). *Let  $\mathbf{M}_1, \mathbf{M}_2$  be  $n \times n$  matrices and  $t \in \mathbb{R}$ . Then we have*

$$e^{(\mathbf{M}_1 + \mathbf{M}_2)t} = e^{\mathbf{M}_1 t} + \int_0^t e^{\mathbf{M}_1(t-s)} \mathbf{M}_2 e^{(\mathbf{M}_1 + \mathbf{M}_2)s} ds.$$

Moreover, if  $\mathbf{A}(t)$  is a matrix-valued function of  $t \in \mathbb{R}$  that is  $C^\infty$  in the sense that each matrix element  $[\mathbf{A}(t)]_{jk}$  is  $C^\infty$ . Then

$$\frac{d e^{\mathbf{A}(t)}}{dt} = \int_0^1 e^{s\mathbf{A}(t)} \mathbf{A}'(t) e^{(1-s)\mathbf{A}(t)} ds.$$

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